Midterm II(A): The Second

N

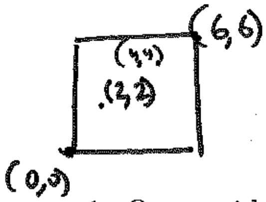
February 27 2017

This test totals 45 points and you get 50 minutes to do it. Answer the questions in the spaces provided on the question sheets. Show work unless the question says otherwise. Good luck!

Name : ______________________________

ID number : ______________________________

Question	Points	Score
1	10	
2	5	
3	5	
4	10	
5	10	
6	5	
Total:	45	



1. On a grid, you are allowed to move only up or right. Let $A = (0,0), B = (6,6)$. Find the number[1] of (shortest) grid paths from A to B which

   (a) (2 points) go through $(2,2)$

   $C(4,2)\ C(8,4)$

   (b) (2 points) go through $(4,4)$

   $C(8,4)\ C(4,2)$

   (c) (2 points) go through $(2,2)$ and $(4,4)$

   $C(4,2)\ C(4,2)\ C(4,2)$

   (d) (2 points) go through $(2,2)$ or $(4,4)$

   $2\ C(4,2)\,C(8,4) - C(4,2)\,C(4,2)\,C(4,2)$

   $= C(4,2)\left[\,2C(8,4) - C(4,2)^2\,\right]$

   (e) (2 points) do not go through $(2,2)$ nor through $(4,4)$.

   $C(12,6) - \left[\,2\,C(4,2)\,C(8,4) - C(4,2)\,C(4,2)\,C(4,2)\,\right]$

   $= C(12,6) + C(4,2)\,C(4,2)\,C(4,2)$

   $\quad - 2C(4,2)\,C(8,4)$

[1] You can leave your answer in binomial coefficients.

2. (5 points) Solve the recurrence $3a_n = 7a_{n-1} + 6a_{n-2}$ with $a_0 = 4$ and $a_1 = 1$.

$$a_n = \frac{7}{3} a_{n-1} + 2a_{n-2}$$

Char eq
$$x^2 - \frac{7}{3}x - 2 = 0$$

$$3x^2 - 7x - 6 = 0$$

$$3x^2 - 9x + 2x - 6 = 0$$

$$3x(x-3) + 2(x-3) = 0$$

$$(3x+2)(x-3) = 0$$

$$\left(x + \frac{2}{3}\right)(x-3) = 0$$

$r_1 = -\frac{2}{3}$ $\quad r_2 = 3$

General sol: $\quad A\left(-\frac{2}{3}\right)^n + B(3)^n = a_n$

$a_0 = 4 = A + B \quad \Rightarrow \quad +12 = 3A + 3B$

$a_1 = 1 = -\frac{2}{3}A + 3B \qquad 1 = -\frac{2}{3}A + 3B$

$+11 = \frac{11}{3}A, \quad A = 3$

$B = 1$

Sol $\quad a_n = 3\left(\frac{-2}{3}\right)^n + 3^n$

3. (5 points) Let $1 \leq k \leq m \leq n$ be integers. Give a *combinatorial* explanation[2] to show

$$C(n, m)C(m, k) = C(n, k)C(n - k, m - k).$$

LHS = $C(n,m)\ C(m,k)$

= choosing m #s from $1, 2, \ldots, n$ and marking k of the numbers amongst the chosen m #s with $*$.

(ex) $n = 5$, $m = 4$, $k = 2$

$C(5,4)\ C(4,2) = \{ \{1, \overset{*}{3}, \overset{*}{4}, 5\}, \{\overset{*}{2}, \overset{*}{3}, 4, 5\}, \ldots \ldots \}$

RHS = $C(n,k)\ C(n-k, m-k)$

= first choosing k #s from $1, 2, \ldots n$ and marking them with a $*$ and then choosing remaining $m-k$ #s from the left over $n-k$ #s.

[2]Please be as precise as possible, points will be taken off if the answer is not coherent/lacks details. If you use the algebraic formula for $C(n, k)$ to show it, you will not get any points for your answer.

4. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

(a) Number of length 5 strings on $\{0,1\}$ with exactly three 1s and no substring 00 is
A. $C(5,2)$ B. $C(5,3)$ C. $C(4,2)$ D. $C(4,3)$ E. None of these.

(b) Number of relations on $X = \{1,2,3,4\}$ which are both reflexive and symmetric is
A. 2^6 B. 2^{16} C. 2^{12} D. 2^4 E. None of these.

(c) $1 - 2C(5,1) + 4C(5,2) - 8C(5,3) + 16C(5,4) - 32$ equals
A. 1 B. -1 C. 0 D. 32 E. None of these.

(d) A simple graph has 6 vertices and 15 edges. The minimum among the degree of its vertices is
A. 6 B. 5 C. 2 D. 0 E. Cannot say.

(e) Number of ways of dividing ten identical candies among 4 children is
A. $C(14,10)$ B. $C(14,3)$ C. $C(13,4)$ D. $C(13,10)$ E. None of these.

Space for scratch work :

(a) 3 1s $\bullet\ 1\ \bullet\ 1\ \bullet\ 1\ \bullet$ 4 spots for 2 0s

$\Rightarrow C(4,2)$

(b) $(i,i) \in R$ for $i = 1,2,3,4.$

$(x,y) \in R \Rightarrow (y,x) \in R$ $\Rightarrow$ $C(4,2)$ ways of picking 2 distinct elements. each $\{x,y\}$ is there in R or not there in R

$\therefore\ 2^{C(4,2)} = 2^{\frac{4\times 3}{2}} = 2^6$

(c) $(1-2)^5 = 1 - 2C(5,1) + 4C(5,2) - 8C(5,3) + 16C(5,4) - 32$

$= -1$

(d) $C(6,2) = \frac{6\times 5}{2} = 15$, $\therefore$ graph is K_6. $\therefore$ min deg $= 5$

(e) $C(10+4-1, 4-1) = C(13,3) = C(13,10)$

5. (10 points) Let a_n denote the number of length n strings on $\{0, 1\}$ which contain the pattern 00. Find a recurrence relation that a_n satisfies. Make sure to also give the initial conditions.

string

begins with 0

0 ______

00 ____

can be any string

$\therefore 2^{n-2}$

01 ____

should contain 00

$\therefore a_{n-2}$

begins with 1

1 ______

should contain 00

$\therefore a_{n-1}$

$$a_n = a_{n-1} + a_{n-2} + 2^{n-2}$$

$$a_1 = 0$$

$$a_2 = 1$$

$$a_3 = 0 + 1 + 2^1 = 3$$

$$\begin{bmatrix} 001 \\ 000 \\ 100 \end{bmatrix}$$

$\vdots$

6. (5 points) Suppose that 7 distinct integers are chosen from the set $\{1, 2, 3, ..., 12\}$. Show that at least two of the six numbers differ[3] by exactly 3. Also give an example to show it is possible to choose the 7 distinct integers such that no two of them differ by exactly 4.

Holes : Boxes labelled (6 holes)

(1, 4) (2, 5) (3, 6) (7, 10)

(8, 11) (9, 12)

Pigeons : 7 chosen #s.

Each # goes into the box to which label it belongs

By PHP, one box has atleast 2 #s which differ by exactly 3.

For 7 #s that never differ by 4

$\{1, 2, 3, 9, 10, 11, 12\}$.

[3]Difference between n and m is $|n - m|$

Midterm II(B): The Second

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Question	Points	Score
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3	10	
4	5	
5	5	
6	10	
Total:	45	

1. (10 points) Let a_n denote the number of length n strings on $\{0, 1\}$ which contain the pattern 11. Find a recurrence relation that a_n satisfies. Make sure to also give the initial conditions.

string

begins with 0

0 ___

should have 11

$\therefore a_{n-1}$

begins with 1

1 ___

11 ___

can be anything

2^{n-2}

10 ___

should have 11

a_{n-2}

$$\therefore \quad a_n = a_{n-1} + a_{n-2} + 2^{n-2}$$

$a_1 = 0$

$a_2 = 1$

$a_3 = 0 + 1 + 2^1 = 3$ $\quad \left\{ \begin{array}{l} 110 \\ 111 \\ 011 \end{array} \right\}$

$\vdots$

2. (5 points) Let $1 \leq k \leq m \leq n$ be integers. Give a *combinatorial* explanation[1] to show

$$C(n,m)C(m,k) = C(n,k)C(n-k,m-k).$$

Check version A

[1]Please be as precise as possible, points will be taken off if the answer is not coherent/lacks details. If you use the algebraic formula for $C(n,k)$ to show it, you will not get any points for your answer.

3. On a grid, you are allowed to move only up or right. Let $A = (0,0), B = (6,6)$. Find the number[2] of (shortest) grid paths from A to B which

(a) (2 points) go through $(3,3)$

$C(6,3)\ C(6,3)$

(b) (2 points) go through $(5,5)$

$C(10,5)\ C(2,1)$

(c) (2 points) go through $(3,3)$ and $(5,5)$

$C(6,3)\ C(4,2)\ C(2,1)$

(d) (2 points) go through $(3,3)$ or $(5,5)$

$C(6,3)^2 + C(10,5)\ C(2,1) - C(6,3)\ C(4,2)\ C(2,1)$

(e) (2 points) do not go through $(3,3)$ nor through $(5,5)$.

$C(12,6) - C(6,3)^2 - C(10,5)\ C(2,1)$

$+ C(6,3)\ C(4,2)\ C(2,1)$

[2]You can leave your answer in binomial coefficients.

4. (5 points) Suppose that 7 distinct integers are chosen from the set $\{1, 2, 3, ..., 12\}$. Show that at least two of the six numbers differ[3] by exactly 3. Also give an example to show it is possible to choose the 7 distinct integers such that no two of them differ by exactly 4.

check version A

[3] Difference between n and m is $|n - m|$

5. (5 points) Solve the recurrence $4a_n = -4a_{n-1} + 3a_{n-2}$ with $a_0 = 0$ and $a_1 = -2$.

char eq: $x^2 + x - \frac{3}{4} = 0$

$4x^2 + 4x - 3 = 0$

$4x^2 + 6x - 2x - 3 = 0$

$2x(2x+3) - 1(2x+3) = 0$

$(2x-1)(2x+3) = 0$

$x = 1/2 \quad x = -3/2$

$r_1 = 1/2, \quad r_2 = -3/2 \qquad r_1 \neq r_2$

general sol

$A\left(\frac{1}{2}\right)^n + B\left(\frac{-3}{2}\right)^n = a_n$

$a_0 = A + B = 0 \quad \Rightarrow A = -B$

$a_1 = \frac{A}{2} - \frac{3B}{2} = -2 \quad \Rightarrow \frac{-B}{2} - \frac{3B}{2} = -2$

$\Rightarrow -4B = -4$

$\Rightarrow B = 1$

$\Rightarrow A = -1$

$$\boxed{a_n = -\left(\frac{1}{2}\right)^n + \left(\frac{-3}{2}\right)^n}$$

6. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

   (a) Number of length 7 strings on $\{0,1\}$ with exactly five 0s and no substring 11 is
   A. C(6, 2) B. C(6, 3) C. C(7, 2) D. C(7, 3) E. None of these.

   (b) Number of relations on $X = \{1, 2, 3, 4\}$ which are both reflexive and symmetric is
   A. 2^{16} B. 2^{12} C. 2^{6} D. 2^{4} E. None of these.

   (c) $-1 + 2C(5,1) - 4C(5,2) + 8C(5,3) - 16C(5,4) + 32$ equals
   A. 1 B. -1 C. 0 D. 32 E. None of these.

   (d) A simple graph has 7 vertices and 21 edges. The minimum among the degree of its vertices is
   A. 6 B. 7 C. 4 D. 0 E. Cannot say.

   (e) Number of ways of dividing ten identical candies among 5 children is
   A. C(14, 5) B. C(14, 4) C. C(15, 4) D. C(15, 10) E. None of these.

Space for scratch work :

c $(2-1)^5 = 1$

D $C(7,2) = \frac{7\times 6}{2} = 21$: graph is K_7.

$\therefore$ min deg $= 6$

e $C(10+5-1,\ 5-1) = C(14, 4) = C(14, 10)$