Midterm II(B): The Second

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> This test totals 45 points and you get 50 minutes to do it. Answer the questions in the spaces provided on the question sheets. Show work unless the question says otherwise. Good luck!

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Question	Points	Score
1	10	10
2	5	5
3	10	10
4	5	5
5	5	5
6	10	10
Total:	45	45

1. (10 points) Let a_n denote the number of length n strings on $\{0, 1\}$ which contain the pattern 11. Find a recurrence relation that a_n satisfies. Make sure to also give the initial conditions.

string

0 __

a_{n-1}

1 __

11 __

It already contains "11", so it's free to put anything in the following string.

2^{n-2} ways.

10 __

a_{n-2}

So $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$

$a_1 = 0$

$a_2 = 1$

Let $1 \le k \le m \le n$ be integers. Give a *combinatorial* explanation[1] to show

$$C(n, m)C(m, k) = C(n, k)C(n-k, m-k).$$

$$\frac{n!}{\cancel{m!}(n-m)!} \frac{m!}{k!(m-k)!}$$

Assume we have to choose m people from n people, and choose k leaders from these m people

It's $C(n, m)\,C(m, k)$

we can divide the m people into 2 groups, k leaders and m-k other people.

we first choose k leaders in n people, and choose the the m-k other people the the n-k people left.

$(n-k)-(m-k)$

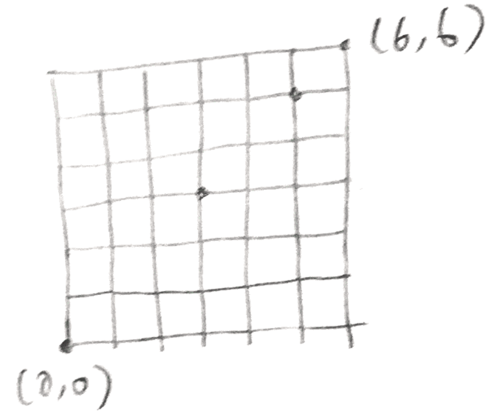
it's $C(n,k)\,C(n-k, m-k) = \frac{n!}{k!\cancel{(n-k)!}} \frac{\cancel{(n-k)!}}{(m-k)!\,(n-m)!}$

$$= \frac{n!}{(n-m)!\,m!} \frac{m!}{k!\,(m-k)!}$$

$$= C(n, m)\,C(m, k).$$

[1]as precise as possible, points will be taken off if the answer is not coherent/lacks details. If ...gebraic formula for $C(n, k)$ to show it, you will not get any points for your answer.

3. On a grid, you are allowed to move only up or right. Let $A = (0,0), B = (6,6)$. Find the number[2] of (shortest) grid paths from A to B which



(a) (2 points) go through $(3,3)$

$C(6,3)\cdot C(6,3) = \binom{6}{3}^2$

2

(b) (2 points) go through $(5,5)$

2 $C(10,5)\cdot C(2,1) = 2\binom{10}{5}$

(c) (2 points) go through $(3,3)$ and $(5,5)$

$C(6,3)\cdot C(4,2)\cdot C(2,1) = 2\binom{6}{3}\binom{4}{2}$

2

(d) (2 points) go through $(3,3)$ or $(5,5)$

$\binom{6}{3}^2 + 2\binom{10}{5} - 2\binom{6}{3}\binom{4}{2}$

2

(e) (2 points) do not go through $(3,3)$ nor through $(5,5)$.

total ways: $C(12,6)$

2 So it's $\binom{12}{6} - \binom{6}{3}^2 - 2\binom{10}{5} + 2\binom{6}{3}\binom{4}{2}$

[2] You can leave your answer in binomial coefficients.

(5 points) Suppose that 7 distinct integers are chosen from the set $\{1, 2, 3, ..., 12\}$. Show that at least two of the ~~six~~ numbers differ[3] by exactly 3. Also give an example to show it is possible to choose the 7 distinct integers such that no two of them differ by exactly 4.

If we divide $\{1, 2, 3 \cdots 12\}$ into 6 sets

$\{1,4\}$, $\{2,5\}$, $\{3,6\}$ $\{7,10\}$, $\{8,11\}$, $\{9,12\}$.

and pick 7 distinct integers and put them into the corresponding set.

There are 7 distinct integers but only 6 sets.

So by pigeonhole principle, there is at least 2 numbers in one set.

Because in each set, the difference between 2 numbers are 3, so there are at least two of 7 numbers differ by exactly 3.

example: 1, 2, 3, 4, 10, 11, 12

none of two of them differ by 4.

[3] Difference between n and m is $|n - m|$

5. (5 points) Solve the recurrence $4a_n = -4a_{n-1} + 3a_{n-2}$ with $a_0 = 0$ and $a_1 = -2$. 2

5

$$4x^2+4x-3=0$$

2 -1

2 3

$$(2x-1)(2x+3)=0$$

$$r_1 = \frac{1}{2}$$

$$r_2 = -\frac{3}{2}$$

since $a_0 = 0$ & $a_1 = -2$

$$a_n = A\left(\frac{1}{2}\right)^n + B\left(-\frac{3}{2}\right)^n$$

$$\begin{cases} 0 = A + B \\ -2 = \frac{1}{2}A - \frac{3}{2}B \end{cases}$$

$-4 = A - 3B$

$$\begin{cases} A = -1 \\ B = 1 \end{cases}$$

So $a_n = -\left(\frac{1}{2}\right)^n + \left(-\frac{3}{2}\right)^n$

6. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

10

0 0 0 0 0

(a) Number of length 7 strings on $\{0,1\}$ with exactly five 0s and no substring 11 is

A. $C(6,2)$ B. $C(6,3)$ C. $C(7,2)$ D. $C(7,3)$ E. None of these.

(b) Number of relations on $X = \{1,2,3,4\}$ which are both reflexive and symmetric is

A. 2^{16} B. 2^{12} C. 2^{6} D. 2^{4} E. None of these.

10 40 80 80

(c) $-1 + 2C(5,1) - 4C(5,2) + 8C(5,3) - 16C(5,4) + 32$ equals $\binom{5}{1}=\binom{5}{4}=5$ $\binom{5}{2}=\binom{5}{3}=10$

A. 1 B. -1 C. 0 D. 32 E. None of these.

(d) A simple graph has 7 vertices and 21 edges. The minimum among the degree of its vertices is 42 degrees.

A. 6 B. 7 C. 4 D. 0 E. Cannot say.

(e) Number of ways of dividing ten identical candies among 5 children is

A. $C(14,5)$ B. $C(14,4)$ C. $C(15,4)$ D. $C(15,10)$ E. None of these.

Space for scratch work :

x x x x x x x x x x

$(10+4, 4)$