Midterm I(A): The First

N

January 30 2017

> This test totals 45 points and you get 50 minutes to do it. Answer the questions in the spaces provided on the question sheets. Show work unless the question says otherwise. Good luck!

Name : ______________________________

ID number : ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	5	
5	10	
Total:	45	

1. (10 points) Using induction, show that 6 divides $n^3 - n$ for all integers $n \geq 1$.

$P(n)$: 6 divides $n^3 - n$

Base case ($n=1$)

$n^3 - n = 1^3 - 1 = 0 = 0 \times 6$

$\therefore$ 6 divides $1^3 - 1$

$\therefore$ $P(1)$ true

Induction hypothesis

Assume $P(N)$ for some integer $N \geq 1$, (ie) $N^3 - N = 6a$ for some $a \in \mathbb{Z}$

Claim: $P(N+1)$ true

$$(N+1)^3 - (N+1) = N^3 + 1 + 3N^2 + 3N - N - 1$$

$$= (N^3 - N) + 3N^2 + 3N$$

$$= 6a + 3(N)(N+1)$$

Note $N(N+1)$ is even (as N or $N+1$ is even!)

$\therefore$ $3N(N+1)$ is a multiple of 6, ie $= 6b$

$$\therefore (N+1)^3 - (N+1) = 6a + 6b$$

$$= 6(a+b)$$

$\therefore$ By induction $P(n)$ true for all $n \geq 1$.

2. A survey was conducted among 24 people about three ice-cream flavours (chocolate, butterscotch and strawberry). It was found that the number of people who liked chocolate was the same as the number of people who liked butterscotch which was also the same as the number of people who liked strawberry. It was also found that 7 people liked none of these flavours, 5 people liked all these flavours, 7 people liked both butterscotch and chocolate, 6 people liked both chocolate and strawberry and 5 people liked both butterscotch and strawberry. Let U represent the set of 24 people surveyed, C represent the set of people who liked chocolate, S the set of people who liked strawberry and B, the set of people who liked butterscotch. Answer the following questions:

No need to show work for this question. However a Venn diagram is given in case you'd like to use it. Also no partial credit will be given for this question. So double check your answers!

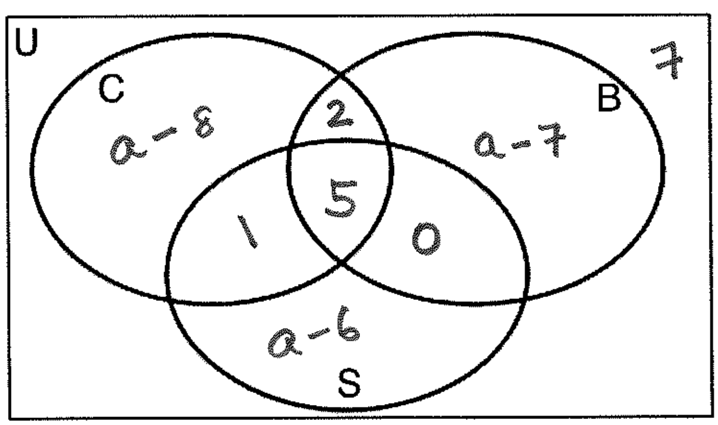
(a) (2 points) What is $|\overline{C \cup B \cup S}|$? 7

(b) (2 points) What is $|C \cap B \cap S|$? 5

(c) (2 points) What is $|S \cap B|$? 5

(d) (2 points) How many people liked chocolate flavour? 10

(e) (2 points) How many people liked chocolate flavour but did not like strawberry nor butterscotch ? 2



$a-8 + a-7 + a-6 + 8+7 = 24$

$\therefore \quad 3a - 13 + 7 = 24$

$3a = 30$

$a = 10$

3. Consider the following relation T defined on $\mathbb{R}$, the set of real numbers.

$$(x, y) \in T \text{ if and only if } x - y \leq 1.$$

Answer the following questions, FULLY JUSTIFYING YOUR ANSWER. If your answer is yes, explain why. If your answer is no, give a counter-example.

(a) (2 points) Is T a function ?

No. $(1,0) \in T$ and $(1, 0.5) \in T$

1 is related to 0 and 0.5 (and many others).

A function cannot have an input with multiple outputs

(b) (2 points) Is T reflexive ?

yes. Any $x \in \mathbb{R}$ has $x - x = 0 \leq 1$

$\therefore (x,x) \in T$ for all $x \in \mathbb{R}$

(c) (2 points) Is T symmetric ?

No. $(0,2) \in T$ as $0-2 = -2 \leq 1$

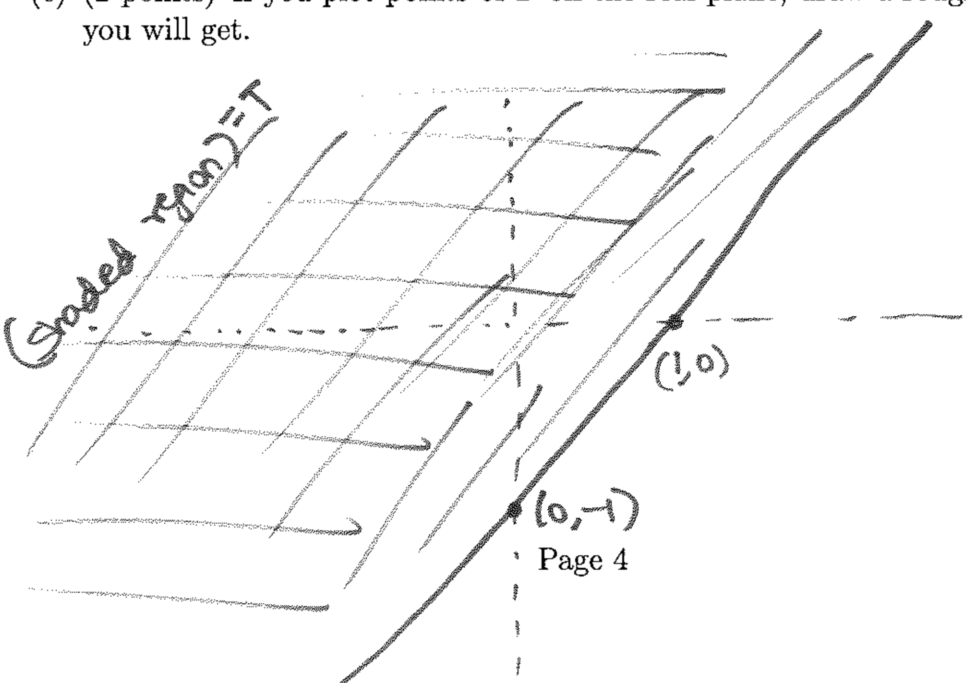
but $(2,0) \notin T$ as $2-0 = 2 > 1$

(d) (2 points) Is T transitive ?

No. if $(x,y) \in T$, $(y,z) \in T \not\Rightarrow (x,z) \in T$

(ex) $(2,1) \in T$ $(1,0) \in T$ but $(2,0) \notin T$

(e) (2 points) If you plot points of T on the real plane, draw a rough picture of what you will get.



4. (5 points) For an integer $n \geq 1$, let $Y = \{1, 2, 3, \ldots, n\}$ and let $P(Y)$ denote the power set of Y. Let $X = \{0, 1\}$ and let Z denote the set of all strings on X of length n. Construct a bijection $f : P(Y) \to Z$ (Explain carefully what the function is and why it is a bijection)

$f : P(Y) \longrightarrow Z$

If $A \subseteq Y$ is a subset, define $f(A)$ to be a string of length n with 1 in the i^{th} place if and only if $i \in A$ and 0 otherwise for all $i \leq n$

Thus $f(\{1\}) = 100 \ldots 0$

$f(\{3,4\}) = 00110 \ldots 0$

1-1 if $f(A) = f(B)$, then $i \in A \Leftrightarrow i \in B$ $\forall i \leq n$

$\therefore A = B$

onto Any string of length n on $\{0,1\}$ comes from a subset

(ie) If s is a string of length n, it is the image of the subset $\{ i \leq n \mid s$ has a 1 in i^{th} place $\}$.

5. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

(a) Number of functions from $\{1,2\}$ to $\{a,b,c\}$
A. 8 B. 9 C. 6 D. ∞ E. None of these.

(b) Number of onto functions from $\{1,2\}$ to $\{a,b,c\}$
A. 6 B. 5 C. 0 D. 2 E. None of these.

(c) Number of one-one functions from $\{1,2\}$ to $\{a,b,c\}$
A. 6 B. 5 C. 0 D. 2 E. None of these.

(d) Number of relations from $\{1,2\}$ to $\{a,b,c\}$
A. 6 B. 2^5 C. ∞ D. 2^6 E. None of these.

(e) Number of equivalence relations on $\{1,2,3\}$
A. 7 B. 5 C. 8 D. ∞ E. None of these.

Space for scratch work :

a) 3 choices for 1, 3 choices for 2

$\therefore 3 \times 3$

b) no onto!

c) 3 choices for 1, 2 choices for 2 $\therefore 3 \times 2$

d) $R \subseteq X \times Y$ $|X| = 2$, $|Y| = 3$ $\therefore$ $|X \times Y|$

$\therefore$ # of relations $= 2^6$

e) # equivalence relations: (1,2,3) (3)(12)

(1)(23) (2)(13)

(1)(2)(3)

Midterm I(B): The First

N

January 30 2017

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Question	Points	Score
1	10	
2	5	
3	10	
4	10	
5	10	
Total:	45	

1. Consider the following relation T defined on $\mathbb{R}$, the set of real numbers.

$$(x, y) \in T \text{ if and only if } x + y \leq 1.$$

Answer the following questions, FULLY JUSTIFYING YOUR ANSWER. If your answer is yes, explain why. If your answer is no, give a counter-example.

(a) (2 points) Is T a function ?

No. as $(0,1) \in T$ $(0, 0.5) \in T$

0 related to both 1, 0.5 (and others)

A function cannot have one input with multiple outputs

(b) (2 points) Is T reflexive ?

No. $3 + 3 \not\leq 1$. $(3,3) \notin T$

$\therefore$ 3 not related to 3

(c) (2 points) Is T symmetric ?

yes. if $(x,y) \in T$, $x + y \leq 1$

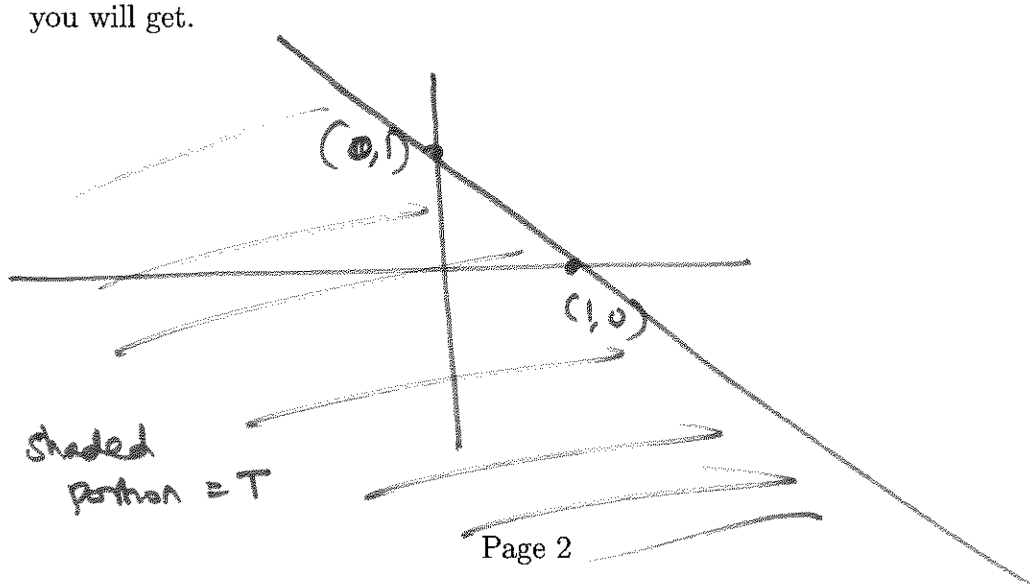
this means $(y+x) \leq 1$ so $(y,x) \in T$

(d) (2 points) Is T transitive ?

No. $(0.8, 0.1) \in T$ $(0.1, 0.7) \in T$

but $(0.8, 0.7) \notin T$ as $0.8 + 0.7 = 1.5 > 1$

(e) (2 points) If you plot points of T on the real plane, draw a rough picture of what you will get.



2. (5 points) For an integer $n \geq 1$, let $Y = \{1, 2, 3, \ldots, n\}$ and let $P(Y)$ denote the power set of Y. Let $X = \{0, 1\}$ and let Z denote the set of all strings on X of length n. Construct a bijection $f : P(Y) \to Z$ (Explain carefully what the function is and why it is a bijection)

(Ref to version A)

3. (10 points) Using induction, show that 6 divides $n^3 - n$ for all integers $n \geq 1$.

(Ref to version A)

5. A survey was conducted among 27 people about three ice-cream flavours (chocolate, butterscotch and strawberry). It was found that the number of people who liked chocolate was the same as the number of people who liked butterscotch which was also the same as the number of people who liked strawberry. It was also found that 2 people liked none of these flavours, 4 people liked all these flavours, 6 people liked both butterscotch and chocolate, 4 people liked both chocolate and strawberry and 5 people liked both butterscotch and strawberry. Let U represent the set of 27 people surveyed, C represent the set of people who liked chocolate, S the set of people who liked strawberry and B, the set of people who liked butterscotch. Answer the following questions:

No need to show work for this question. However a Venn diagram is given in case you'd like to use it. Also no partial credit will be given for this question. So double check your answers!

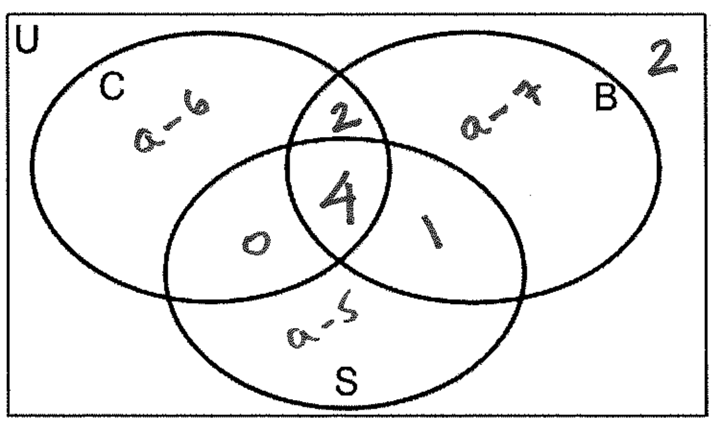
(a) (2 points) What is $|\overline{C \cup B \cup S}|$? 2

(b) (2 points) What is $|C \cap B \cap S|$? 4

(c) (2 points) What is $|S \cap B|$? 5

(d) (2 points) How many people liked chocolate flavour? 12

(e) (2 points) How many people liked chocolate flavour but did not like strawberry nor butterscotch ? 6



$a-6 + a-7 + a-5 + 7 + 2 = 27$

$3a - 9 = 27$

$3a = 36$

$a = 12$

4. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

   (a) Number of functions from $\{a, b, c\}$ to $\{1, 2\}$
   A. 8 B. 9 C. 6 D. ∞ E. None of these.

   (b) Number of onto functions from $\{a, b, c\}$ to $\{1, 2, 3\}$
   A. 6 B. 5 C. 0 D. 2 E. None of these.

   (c) Number of one-one functions from $\{a, b, c\}$ to $\{1, 2\}$
   A. 6 B. 5 C. 0 D. 2 E. None of these.

   (d) Number of relations from $\{a, b, c\}$ to $\{1, 2\}$
   A. 6 B. 2^5 C. ∞ D. 2^6 E. None of these.

   (e) Number of equivalence relations on $\{1, 2\}$
   A. 7 B. 5 C. 2 D. ∞ E. None of these.

Space for scratch work :

(a) 2 choices for a, 2 choices for b,
2 choices for c. $\therefore$ $2 \times 2 \times 2$

(b) # of onto = # of 1-1 functions from
$\{a, b, c\}$ to $\{1, 2, 3\}$ = $3 \times 2 \times 1 = 6$

(c) # of 1-1 = 0 !

(d) $X = \{a, b, c\}$, $Y = \{1, 2\}$ $X \times Y$ has 6 elt

$\therefore$ # of relations = 2^6

(e) # of eq relations (1) (2) (12)