1. Consider the following relation T defined on $\mathbb{R}$, the set of real numbers.

$$(x, y) \in T \text{ if and only if } x + y \leq 1.$$

Answer the following questions, FULLY JUSTIFYING YOUR ANSWER. If your answer is yes, explain why. If your answer is no, give a counter-example.

(a) (2 points) Is T a function ?

No, since $(3,-3) \in T$ & $(3,-4) \in T$, meaning

2 that domains have more than one value associated with it.

(b) (2 points) Is T reflexive ?

No. $(-3,2) \in T$ since $-3+2 \leq 1$, but

2 $(2,2) \notin T$ since $2+2 > 1$.

(c) (2 points) Is T symmetric ?

Yes. This is because $x+y = y+x$, which will be ≤ 1 if $x+y \leq 1$.

2 ex) $(-3,2) \in T$ since $-3+2 \leq 1$, $(2,-3) \in T$ since $2+(-3) \leq 1$.

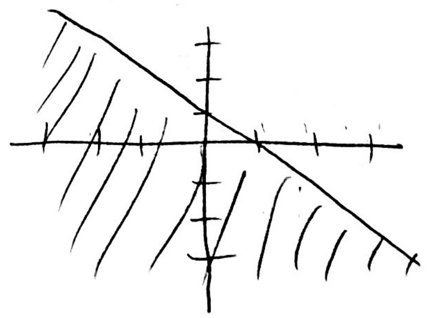
(d) (2 points) Is T transitive ?

Yes. This is because whenever $(x,y) \in T$ and $(y,z) \in T$, $(x,z) \in T$

0 since if $x+y \leq 1$ & $y+z \leq 1$, $y \leq 1-z$

(e) (2 points) If you plot points of T on the real plane, draw a rough graph of what you will get.

2



2. (5 points) For an integer $n \geq 1$, let $Y = \{1, 2, 3, \ldots, n\}$ and let $P(Y)$ denote the power set of Y. Let $X = \{0, 1\}$ and let Z denote the set of all strings on X of length n. Construct a bijection $f : P(Y) \to Z$ (Explain carefully what the function is and why it is a bijection)

$P(Y)$ = power sets of Y

$P(1) = \{\{\emptyset\}, \{1\}\}$

$Z = \{0, 1\}$

$P(2) = \{\{\emptyset\}, \{1\}, \{2\}, \{1,2\}\}$

$Z = \{\{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}\}$

$\emptyset \to \{0,0\}$

$1 \to \{0,1\}$

$2 \to \{1,0\}$

$1,2 \to \{1,1\}$

$P(3) = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$Z = \{\{0,0,0\}, \{0,0,1\}, \{0,1,0\}, \{0,1,1\}, \{1,0,0\}, \{1,0,1\},$

$f : P(Y) \to Z$

The function is that whenever there is a "1" in the subset The last digit in the string is a "1", else, it is a zero. When there is a "2" in the subset, there is a "1" in the second to last digit and so on. Essentially, the subset gives the location of the "1"'s in the string. This is one-to-one since $|P(Y)| = |Z|$, and each subset has a different string associated with it. Likewise, since it is one-to-one and $|P(Y)| = |Z|$, every value in the codomain is used, meaning that the range is the codomain.

3. (10 points) Using induction, show that 6 divides $n^3 - n$ for all integers $n \geq 1$.

$P(n)$: $n^3 - n$ is divisible by 6 $\quad n \geq 1$

$P(b)$: $P(1)$: $1^3 - 1 = 0$ is divisible by 6 ✓

Induction hypothesis: $n^3 - n$ is divisible by 6

Show that $(n+1)^3 - (n+1)$ is divisible by 6

$(n^2+2n+1)(n+1) - (n+1)$

$n^3 + n^2 + 2n^2 + 2n + n + 1 - (n+1)$

$n^3 + 3n^2 + 3n + 1 - n - 1$

$n^3 + 3n^2 + 2n$ → use induction to prove this too.

$n^3 - n + \overbrace{3n^2 + 3n}$

By induction hypothesis, $n^3 - n$ is divisible by 6.

Show $3n^2 + 3n$ is too $\quad n \geq 1$

$P(b)$: $P(1)$: $3 + 3 = 6$ is divisible by 6 ✓

Induction hypothesis$_2$: $3n^2 + 3n$ is divisible by 6

$3(n+1)^2 + 3(n+1)$

$3(n^2+2n+1) + 3n + 3$

$3n^2 + 6n + 3 + 3n + 3$

$3n^2 + 3n + 6n + 6 = 3n^2 + 3n + 6(n+6)$

By induction hypothesis$_2$, $3n^2 + 3n$ is divisible by 6.

It is also clear that $6(n+6)$ is divisible by 6.

Thus, $3n^2 + 3n + 6n + 6$ is divisible by 6, meaning that $3n^2 + 3n$ is divisible by 6, which again shows that $n^3 - n + 3n^2 + 3n$ is divisible by 6, which finally shows $n^3 - n$ is divisible by 6.

4. (10 points) Circle the correct answer. No need to show work for this question. (Also no partial credit)

$2^3 = 8$

(a) Number of functions from $\{a, b, c\}$ to $\{1, 2\}$

A. 8 B. 9 C. 6 D. ∞ E. None of these.

(b) Number of onto functions from $\{a, b, c\}$ to $\{1, 2, 3\}$ $3 \times 2 = 6$

A. 6 B. 5 C. 0 D. 2 E. None of these.

(c) Number of one-one functions from $\{a, b, c\}$ to $\{1, 2\}$

8

A. 6 B. 5 C. 0 D. 2 E. None of these.

(d) Number of relations from $\{a, b, c\}$ to $\{1, 2\}$

A. 6 B. 2^5 C. ∞ D. 2^6 E. None of these.

(e) Number of equivalence relations on $\{1, 2\}$

A. 7 B. 5 C. 2 D. ∞ E. None of these.

Space for scratch work :

5. A survey was conducted among 27 people about three ice-cream flavours (chocolate, butterscotch and strawberry). It was found that the number of people who liked chocolate was the same as the number of people who liked butterscotch which was also the same as the number of people who liked strawberry. It was also found that 2 people liked none of these flavours, 4 people liked all these flavours, 6 people liked both butterscotch and chocolate, 4 people liked both chocolate and strawberry and 5 people liked both butterscotch and strawberry. Let U represent the set of 27 people surveyed, C represent the set of people who liked chocolate, S the set of people who liked strawberry and B, the set of people who liked butterscotch. Answer the following questions:

No need to show work for this question. However a Venn diagram is given in case you'd like to use it. Also no partial credit will be given for this question. So double check your answers!

2 none
4 CBS
6 CB
4 CS
5 BS

27 people

8

(a) (2 points) What is $|\overline{C \cup B \cup S}|$?

2

(b) (2 points) What is $|C \cap B \cap S|$?

4

(c) (2 points) What is $|S \cap B|$?

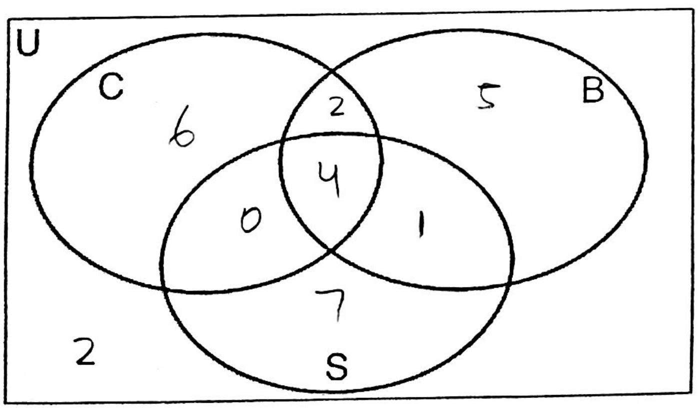
1

(d) (2 points) How many people liked chocolate flavour?

12

(e) (2 points) How many people liked chocolate flavour but did not like strawberry nor butterscotch ?

6



$2+4+1+B$

$7+B = 6+18-B-S$

$2B = 17-S$

$2+4+1+B$
$= 2+4+C$
$= S+S$

$7+B = 6+C = 5+S$

$C+B+S+7 = 27$

$C+B+S = 18$

$C = 18-B-S$