Math 61 **Midterm 1**
Winter 2010

Name: ______________________________

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<u>Instructions:</u>

- **There are 8 problems. Make sure you are not missing any problems.**
- **Explain your answers using complete sentences. Writing a number alone is not enough to earn full credit.**
- **No calculators, books, or notes are allowed.**
- **Do not use your own scratch paper.**

1 BLUE EXAM

1. **(10 points) Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of nonnegative integers. Define the relation R as follows:**

$$R = \{(j, k) \in \mathbb{N} \times \mathbb{N} \big| \text{ the last digit of } j \text{ is equal to the last digit of } k \ \}.$$

Is R a partial order? Is R an equivalence relation? Prove your answers.

SOLUTION : We can say already that R is not a partial order because $(1, 11) \in R$ and $(11, 1) \in R$, so R is not antisymmetric.
R is an equivalence relation. To prove this, we need to check that R is reflexive, transitive, and symmetric.
R **is reflexive.** Proof: Of course $(j, j) \in R$ because the last digit of j is equal to the last digit of j.
R **is symmetric.** Proof: If $(j, k) \in R$, then the last digit of j is equal to the last digit of k. Of course this is the same as saying that the last digit of k is equal to the last digit of j; in other words, $(k, j) \in R$.
R **is transitive.** Proof: Suppose $(j, k) \in R$ and $(k, n) \in R$. Then the last digit of j is equal to the last digit of k, which is equal to the last digit of n. Hence $(j, n) \in R$.

2. (5 points) Suppose the local diner offers 25 soups, 20 salads, and 10 drinks. The lunch combo comes with EITHER one soup, one salad, and one drink, OR two soups and one drink. How many different combos can you choose?

   SOLUTION : By the multiplication principle, there are $25 \cdot 20 \cdot 10$ ways to choose one soup, one salad, and one drink. Also by the multiplication principle, there are $C(25, 2) \cdot 10$ ways to choose choose two different soups and one drink. (Note we use combinations instead of permutations because the ordering of the soups is irrelevant.) So by the addition principle, there are

$$25 \cdot 20 \cdot 10 + C(25, 2) \cdot 10$$

   different combos to choose from.

3. (10 points) The casino has an endless supply of \$3 and \$7 tokens. Prove it is possible, using only these tokens, to place a bet for any dollar amount greater than or equal to \$12 .
   SOLUTION : First we show explicitly that it is possible to place a bet of \$12, \$13, or \$14:

$$12 = 3 + 3 + 3 + 3 \quad (1)$$
$$13 = 3 + 3 + 7 \quad (2)$$
$$14 = 7 + 7. \quad (3)$$

   Note that every integer $n \geq 14$ such that $n = 2 \mod 3$ can be obtained by adding 3's to 14. Similarly, every integer $n \geq 12$ such that $n = 0 \mod 3$ can be obtained by adding 3's to 12. Finally, every integer $n \geq 13$ such that $n = 1 \mod 3$ can be obtained by adding 3's to 13. Since every integer is equal $\mod 3$ to either $0, 1,$ or 2, we have proven the claim for every integer $n \geq 12$.

4. (5 points) Using a standard deck of 52 cards (4 suits, 13 cards each), how many ways are there to draw six cards so that two cards come from one suit, two cards come from another suit, and two cards come from yet another suit?

   SOLUTION : Given a particular suit, there are $C(13,2)$ ways to choose two cards from any given suit. Since there are $C(4,3)$ ways to choose the two different suits, there are

$$C(4,3) \cdot C(13,2) \cdot C(13,2) \cdot C(13,2)$$

   ways of choosing the cards.

5. (5 points) You have 17 identical pieces of candy. You must divide all of the candy among your six best friends. You may give each friend any amount of candy between 0 and 17 pieces (including 0 and 17), but you must give all 17 pieces away. In how many ways can you do this?

   SOLUTION : There are 17 pieces of candy to split into 6 different piles. This means that we need 5 dividing walls to separate the piles. So there are 22 objects – 17 pieces of candy and 5 walls. There are $C(22,5)$ ways to decide where to put the walls. This is the answer.

6. (10 points) Let $X = \{1, 2, 3, \ldots, 100\}$. How many subsets of X do not contain either the element 1 or the element 2?
SOLUTION : Let $Y = \{3, 4, 5, \ldots, 100\}$. Every subset of X that does not contain either 1 or 2 is in one-to-one correspondence with a subset of Y. We know that there are 2^{98} subsets of Y, so there are 2^{98} subsets of X that do not contain either the element 1 or the element 2.

7. (10 points) Let $X = \{1, 2, 3, 4, \ldots, 699, 700\}$. Let $A = \{a \in X \big| a \text{ is divisible by } 5\}$. Let $B = \{a \in X \big| b \text{ is divisible by } 7\}$. How many elements are in $A \cup B$?
SOLUTION : Recall that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

First, we know that A has 140 elements because every 5th integer is divisible by 5, and because $\frac{700}{5} = 140$. Similarly, B has 100 elements because every 7th integer is divisible by 7, and because $\frac{700}{7} = 100$. Also,

$$A \cap B = \{n \in X \big| n \text{ is divisible by } 35\},$$

so $|A \cap B| = \frac{700}{35} = 20$. Hence

$$|A \cup B| = |A| + |B| - |A \cap B| = 140 + 100 - 20 = 220.$$

8. (10 points) If $x_1, x_2, \ldots, x_n$ are real numbers in the interval $[0, 1]$, prove that

$$\prod_{j=1}^{n}(1 - x_i) \geq 1 - \sum_{j=1}^{n} x_j.$$

SOLUTION : We prove this by induction. First, if $n = 1$, this is obvious, since of course $1 - x_1 \geq 1 - x_1$. Now assume that for some n, we know

$$\prod_{j=1}^{n}(1 - x_j) \geq 1 - \sum_{j=1}^{n} x_j.$$

(This is the induction hypothesis.) We must now prove

$$\prod_{j=1}^{n+1}(1 - x_j) \geq 1 - \sum_{j=1}^{n+1} x_j.$$

Note that by the induction hypothesis (which is used in the first inequality below),

$$\begin{aligned}
\prod_{j=1}^{n+1}(1 - x_j) &= (1 - x_{n+1}) \prod_{j=1}^{n}(1 - x_j) && (4)\\
&\geq (1 - x_{n+1}) \left(1 - \sum_{j=1}^{n} x_j\right) && (5)\\
&= 1 - \sum_{j=1}^{n} x_j - x_{n+1} + x_{n+1} \left(\sum_{j=1}^{n} x_j\right) && (6)\\
&\geq 1 - \sum_{j=1}^{n} x_j - x_{n+1} = 1 - \sum_{j=1}^{n+1} x_j, && (7)
\end{aligned}$$

which is what we wanted to prove. The last inequality holds because

$$x_{n+1} \left(\sum_{j=1}^{n} x_j\right) \geq 0.$$

2 YELLOW EXAM

1. **(5 points) Suppose the local diner offers 17 soups, 22 salads, and 10 drinks. The lunch combo comes with EITHER one soup, one salad, and one drink, OR three different soups. How many different combos can you choose?**

   SOLUTION : By the multiplication principle, there are $17 \cdot 22 \cdot 10$ ways to choose one soup, one salad, and one drink. Also by the multiplication principle, there are $C(17, 3)$ ways to choose choose three different soups. (Note we use combinations instead of permutations because the ordering of the soups is irrelevant.) So by the addition principle, there are

   $$17 \cdot 22 \cdot 10 + C(17, 3)$$

   different combos to choose from.

2. (10 points) The casino has an endless supply of \$3 and \$8 tokens. Prove it is possible, using only these tokens, to place a bet for any dollar amount greater than or equal to \$14 .
   SOLUTION : First we show explicitly that it is possible to place a bet of \$14, \$15, or \$16:

   $$14 = 8 + 3 + 3 \quad (8)$$
   $$15 = 3 + 3 + 3 + 3 + 3 \quad (9)$$
   $$16 = 8 + 8. \quad (10)$$

   Note that every integer $n \geq 14$ such that $n = 2 \mod 3$ can be obtained by adding 3's to 14. Similarly, every integer $n \geq 15$ such that $n = 0 \mod 3$ can be obtained by adding 3's to 15. Finally, every integer $n \geq 16$ such that $n = 1 \mod 3$ can be obtained by adding 3's to 16. Since every integer is equal $\mod 3$ to either $0, 1$, or 2, we have proven the claim for every integer $n \geq 14$.

3. (10 points) Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of nonnegative integers. Define the relation R as follows:

$$R = \{(j,k) \in \mathbb{N} \times \mathbb{N} \big| \text{ the last digit of } j \text{ is equal to the last digit of } k \ \}.$$

Is R a partial order? Is R an equivalence relation? Prove your answers.

SOLUTION : We can say already that R is not a partial order because $(1, 11) \in R$ and $(11, 1) \in R$, so R is not antisymmetric.
R is an equivalence relation. To prove this, we need to check that R is reflexive, transitive, and symmetric.
R **is reflexive.** Proof: Of course $(j, j) \in R$ because the last digit of j is equal to the last digit of j.
R **is symmetric.** Proof: If $(j, k) \in R$, then the last digit of j is equal to the last digit of k. Of course this is the same as saying that the last digit of k is equal to the last digit of j; in other words, $(k, j) \in R$.
R **is transitive.** Proof: Suppose $(j, k) \in R$ and $(k, n) \in R$. Then the last digit of j is equal to the last digit of k, which is equal to the last digit of n. Hence $(j, n) \in R$.

4. (5 points) Using a standard deck of 52 cards (4 suits, 13 cards each), how many ways are there to draw six cards so that three cards come from one suit and three cards come from a different suit?
SOLUTION : Given a particular suit, there are $C(13,3)$ ways to choose three cards from any given suit. Since there are $C(4,2)$ ways to choose the two different suits, there are
$$C(4,2) \cdot C(13,3) \cdot C(13,3)$$
ways of choosing the cards.

5. (5 points) You have 23 identical pieces of candy. You must divide all of the candy among your six best friends. You may give each friend any amount of candy between 0 and 23 pieces (including 0 and 23), but you must give all 23 pieces away. In how many ways can you do this?
SOLUTION :
There are 23 pieces of candy to split into 6 different piles. This means that we need 5 dividing walls to separate the piles. So there are 28 objects – 23 pieces of candy and 5 walls. There are $C(28,5)$ ways to decide where to put the walls. This is the answer.

6. (10 points) Let $X = \{1, 2, 3, \ldots, 100\}$. How many subsets of X do not contain either the element 1 or the element 2?
SOLUTION : Let $Y = \{3, 4, 5, \ldots, 100\}$. Every subset of X that does not contain either 1 or 2 is in one-to-one correspondence with a subset of Y. We know that there are 2^{98} subsets of Y, so there are 2^{98} subsets of X that do not contain either the element 1 or the element 2.

7. (10 points) Let $X = \{1, 2, 3, 4, \ldots, 899, 900\}$. Let $A = \{a \in X \big| a \text{ is divisible by } 5\}$. Let $B = \{b \in X \big| b \text{ is divisible by } 9\}$. How many elements are in $A \cup B$?
SOLUTION : Recall that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

First, we know that A has 180 elements because every 5th integer is divisible by 5, and because $\frac{900}{5} = 180$. Similarly, B has 100 elements because every 9th integer is divisible by 9, and because $\frac{900}{9} = 100$. Also,

$$A \cap B = \{n \in X \big| n \text{ is divisible by } 45\},$$

so $|A \cap B| = \frac{900}{45} = 20$. Hence

$$|A \cup B| = |A| + |B| - |A \cap B| = 180 + 100 - 20 = 260.$$

8. (10 points) If $x_1, x_2, \ldots, x_n$ are real numbers in the interval $[0,1]$, prove that

$$\prod_{j=1}^{n}(1-x_j) \geq 1 - \sum_{j=1}^{n} x_j.$$

SOLUTION : We prove this by induction. First, if $n = 1$, this is obvious, since of course $1 - x_1 \geq 1 - x_1$. Now assume that for some n, we know

$$\prod_{j=1}^{n}(1-x_j) \geq 1 - \sum_{j=1}^{n} x_j.$$

(This is the induction hypothesis.) We must now prove

$$\prod_{j=1}^{n+1}(1-x_j) \geq 1 - \sum_{j=1}^{n+1} x_j.$$

Note that by the induction hypothesis,

$$\begin{aligned}
\prod_{j=1}^{n+1}(1-x_j) &= (1-x_{n+1})\prod_{j=1}^{n}(1-x_j) && (11)\\
&\geq (1-x_{n+1})\left(1-\sum_{j=1}^{n} x_j\right) && (12)\\
&= 1-\sum_{j=1}^{n} x_j - x_{n+1} + x_{n+1}\left(\sum_{j=1}^{n} x_j\right) && (13)\\
&\geq 1-\sum_{j=1}^{n} x_j - x_{n+1} = 1 - \sum_{j=1}^{n+1} x_j, && (14)
\end{aligned}$$

which is what we wanted to prove.