Math 61
Version 2
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Midterm 1

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Section 1A

Instructions:

- There are 8 problems. Make sure you are not missing any problems.
- Explain your answers using complete sentences. Writing a number alone is not enough to earn full credit.
- No calculators, books, or notes are allowed.
- Do not use your own scratch paper.

Question	Points	Score
1	5	5
2	5	5
3	10	10
4	15	11
5	10	2
6	10	10
7	10	10
Total:	65	53

1. **(5 points) How many ways are there to form distinct strings using the letters in the word "differentiation"?**

$$\frac{15!}{3!2!2!2!2!}$$

d 1
i 3
f 2
e 2
r 1
n 2
t 2
a 1
o 1

There are 15! ways to arrange the 15 letters of "differentiation". However, there are 3 identical "i's", 2 "f's", 2 "e's", 2 "n's", and 2 "t's". So the number of ways to form distinct strings is

$$\boxed{\frac{15!}{3!2!2!2!2!}}$$

+5

2. **(5 points) A clothing store offers 5 different shirts, 8 different hats, and 13 different jackets. Suppose you need to buy EITHER one shirt, one hat, and one jacket, OR two** *different* shirts and one hat. In how many different ways can you do this?

union

$C(5,1)C(8,1)C(13,1) + C(5,2)C(8,1)C(13,0)$

There are $C(5,1)$ ways to pick one shirt,
$C(8,1)$ ways to pick one hat,
$C(13,1)$ " " " " jacket.
There are $C(5,2)$ " " " two shirts
$C(8,1)$ " " " one hat
$C(13,0)$ " " " no jackets

Since you can pick from either option, the total number of ways to pick is to add the ways to for each option

resulting in $\boxed{C(5,1)C(8,1)C(13,1) + C(5,2)C(8,1)C(13,0)}$ ways

+5

3. **(10 points) Let N be the set of nonnegative integers; i.e., N = $\{0, 1, 2, 3, \dots\}$. Recall that $\mathcal{P}(\mathrm{N})$ is the set of all subsets of N. Define a relation R on $\mathcal{P}(\mathrm{N})$ as follows: we say $(A, B) \in R$ if and only if $A \subseteq B$. Is R an equivalence relation? Is R a partial order? Prove both of your answers.**

R on P(N)

(A, B) ∈ R if A ⊆ B

for all A ∈ P(N), A ⊆ A

for all A ∈ P(N), A is a subset of itself

so (A, A) ∈ R for all A ∈ P(N)

so R is reflexive

if (A, B) ∈ R, then A ⊆ B

it is not always true that B ⊆ A, so for every (A, B) ∈ R there isn't always a (B, A) ∈ R so R is <u>not</u> symmetric

For every (A, B) ∈ R, (B, A) ∈ R if and only if A = B.

(if A ⊆ B, then B ⊆ A only if A = B)

so R is antisymmetric

if (A, B), (B, C) ∈ R then A ⊆ B and B ⊆ C.

If A ⊆ B & B ⊆ C then A ⊆ C so (A, C) ∈ R for all A, B, C ∈ P(N). So R is transitive.

R is a partial order b/c it is reflexive, antisymmetric, and transitive

R is <u>not</u> a equivalence relation b/c it is <u>not</u> symmetric.

+10

4. **(15 points) Let A and B be finite sets. Let $f \colon A \to B$. (This means f is a function from A to B.)**
(A) (5 points) Define a relation R on A as follows: $(x, y) \in R$ if and only if $f(x) = f(y)$. Prove R is an equivalence relation.
(B) (10 points) Assume f is one-to-one and onto. Prove that A and B contain the same number of elements.

A. $(x,y) \in R$ if & only if $f(x)=f(y)$

For all $x \in A$, $f(x) = f(x)$, so $(x,x) \in R$ for all $x \in A$ and R is reflexive

If $(x,y) \in R$, then $f(x)=f(y)$. so $f(y) = f(x)$ also. Therefore $(y,x) \in R$ for every $(x,y) \in R$. So R is symmetric

if (x,y), $(y,z) \in R$. then $f(x)=f(y)$ and $f(y)=f(z)$ therefore $f(x) = f(z)$. So $(x,z) \in R$ also for every (x,y), $(y,z) \in R$. So R is transitive

So R is a equivalence relation

B. If f is one-to-one & $f: A \to B$, every element in B is related to at most one element in A. If f is onto, every element in B is related to at least one element in A. Therefore, every element in B is related to exactly one element in A. Because f is a function, no one element in A can relate to two different elements in B. Therefore A & B ~~must~~ contain the same number of elements.

6/10

explain more.

5. **(10 points)** Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $Y = \{1, 2, 3, 4, 5\}$.
   (A) (5 points) How many distinct functions are there from X to Y?
   (B) (5 points) How many distinct one-to-one functions are there from X to Y?

can a function be empty?

A. $X \begin{pmatrix} 1 \\ 2 \\ \vdots \\ 10 \end{pmatrix}$ $\underline{6}\ \underline{6}\ \underline{6}\ \_\ \_\ \_\ \_\ \_\ \_\ \underline{6}$

$\boxed{6^{10} \text{ distinct functions}}$

8/5

for each element in X, there are five elements in Y it could relate to ~~or~~ it could not relate to any element in Y. So there are $6 \cdot 6 \cdot 6 \cdots 6^n = 6^{10}$ ways to do this

B. $\underline{6}\ \underline{5}\ \underline{4}\ \_\ \_\ \_\ \underline{1}\ \underline{1}\ \underline{1}\ \_\_$ $5!\,6!$

10 $\frac{10!}{5!\,5!}$

$C(10, 5)$

$6!\ 5!$

to be one-to-one, each element in Y must be related to at most one element in X.

So ~~once an element in~~ X is related to an element in Y, no other element in X can relate to the same element in Y.

The first element in X can choose from 5 elements in Y or no elements, The second element can choose from 5 total possibilities, and so on.

There are $10 - 6 + 1$ ways to rearrange the order the elements in X choose the elements in Y. So there are

$\boxed{6!\,5! \text{ distinct one-to-one functions.}}$

6. **(10 points) Let $n \geq 1$ be an integer.**
(A) Prove

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

by using the binomial theorem. (3 points) (Remember that $\sum_{k=0}^{n} C(n,k) =$ $C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n)$.)
(B) Prove the same statement WITHOUT using the binomial theorem. (7 points)

A. binomial thm

$$(a+b)^n = \sum_{k=0}^{n} a^{n-k} \, b^k \, C(n,k)$$

let $a = b = 1$

$$(1+1)^n = \sum_{k=0}^{n} 1^{n-k} \, 1^k \, C(n,k) \qquad +3$$

$$2^n = \sum_{k=0}^{n} C(n,k) \qquad \square$$

B. Sps we have 2 distinct items and we have n of both.
we want to arrange them in n spots.
There are 2^n ways to do this

$$\underset{1}{\underline{2}} \cdot \underline{2} \cdot \underline{2} \;\cdots\; \underline{2} \cdot \underset{n}{\underline{2}}$$

We could also count the ways of doing this by adding up the number of ways we can arrange the first element.
we could choose 0 out of n, 1 out of n ... n out of n

or $C(n,0) + C(n,1) + \cdots + C(n,n)$

This is equal to $\sum_{k=0}^{n} C(n,k)$ and would also give us the number of ways to arrange these two items in n spots.

$$\text{so } 2^n = \sum_{k=0}^{n} C(n,k) \qquad \square \qquad +7$$

7. (10 points) If $x_1, x_2, \ldots, x_n$ are real numbers in the interval $[0, 1]$, prove that

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) \geq 1 - x_1 - x_2 - \cdots - x_n.$$

base: let $n = p$

$(1 - x_p) \geq 1 - x_p$ ✓ $+3$

induction: Assume $(1-x_1)(1-x_2) \cdots (1-x_n) \geq 1 - x_1 - x_2 - \cdots - x_n$

is true for n

Prove for $n+1$

$(1-x_1)(1-x_2) \cdots (1-x_n)(1-x_{n+1}) \geq (1-x_{n+1})(1-x_1-x_2-\cdots-x_n)$

$(1-x_{n+1})$ is never negative
b/c $x \in [0,1]$

$(1-x_{n+1})(1-x_1-x_2-\cdots-x_n) \geq (1-x_1-x_2-\cdots-x_n) - x_{n+1}(1-x_1 \cdots -x$

positive

$\geq$ [illegible]

$= (1-x_{n+1})$ b/c $-x_{n+1}(1-x_1 \cdots - x_n) \geq -x_{n+1}$

b/c $x_1, \ldots x_{n+1} \in [0,1]$

so $(1-x_1)(1-x_2) \cdots (1-x_n)(1-x_{n+1}) \geq 1 - x_1 \cdots - x_n - x_{n+1}$

$\boxed{4}$

$+7$