Math 61 Midterm 1
Version 1
Fall 2011

Name: ____________________

Signature: ____________________

UCLA ID Number: ____________________

Instructions:

- There are 8 problems. Make sure you are not missing any problems.
- Explain your answers using complete sentences. Writing a number alone is not enough to earn full credit.
- No calculators, books, or notes are allowed.
- Do not use your own scratch paper.

Question	Points	Score
1	5	
2	5	
3	10	
4	15	
5	10	
6	10	
7	10	
Total:	65	

1. **(5 points) How many ways are there to form distinct strings using the letters in the word "differentiate"?**

   There are 13 total letters, 2 each of f, i, and t, and 3 of e. Using the notation from class, we have $n_1 = 3$, $n_2 = n_3 = n_4 = 2$, and the rest of the n_j are equal to 1, with $N = 13$. This gives us the answer of

$$\frac{13!}{3!2!2!2!}.$$

2. **(5 points) A clothing store offers 7 different shirts, 9 different hats, and 11 different jackets. Suppose you need to buy EITHER one shirt, one hat, and one jacket, OR two** *different* shirts and one hat. In how many different ways can you do this?

   There are two dijoint options, so we figure out how many ways there are for each option and then add them together. In the first option, we get one of each: using the multiplication principle, there are $7 \cdot 9 \cdot 11$ ways to do this. For the second option, we get two different shirts and one hat: there are C(5,2) ways to choose the shirts and 9 ways to choose the hat, for a total of $C(5, 2) \cdot 9$ ways.

   So the final answer is $7 \cdot 9 \cdot 11 + C(5, 2) \cdot 9$.

3. **(10 points) Let N be the set of nonnegative integers; i.e., $\mathbf{N} = \{0, 1, 2, 3, \dots\}$. Recall that $\mathcal{P}(\mathbf{N})$ is the set of all subsets of N. Define a relation R on $\mathcal{P}(\mathbf{N})$ as follows: we say $(A, B) \in R$ if and only if $A \subseteq B$. Is R an equivalence relation? Is R a partial order? Prove both of your answers.**

R is not an equivalence relation. To see this, we only need to show that R is not symmetric. To see that R is not symmetric, consider the two sets $C = \{0\}$ and $D = \{0, 1\}$. Obviously $C \subseteq D$, but $D \nsubseteq C$ since $1 \in D$ but $1 \notin C$. This means $(C, D) \in R$ but $(D, C) \notin R$.

R is a partial order. To see that R is reflexive, just note that $A \subseteq A$ for any set A; in other words, $(A, A) \in R$.

To see that R is antisymmetric, assume we have two sets A and B with $(A, B) \in R$ and $(B, A) \in R$. By the definition of R, this means $A \subseteq B$ and $B \subseteq A$. But this is exactly what it means for two sets to be equal; in other words, this means $A = B$. This proves antisymmetry.

To see that R is transitive, assume $(A, B) \in R$ and $(B, C) \in R$. This means $A \subseteq B$ and $B \subseteq C$. But if every element of A is in B, and every element of B is in C, then every element of A is in C, proving that $A \subseteq C$. This means that $(A, C) \in R$, proving transitivity.

4. **(15 points) Let A and B be finite sets. Let $f\colon A \to B$. (This means f is a function from A to B.)**
(A) (5 points) Define a relation R on A as follows: $(x, y) \in R$ if and only if $f(x) = f(y)$. Prove R is an equivalence relation.
(B) (10 points) Assume f is one-to-one and onto. Prove that A and B contain the same number of elements.

(A). R is reflexive because of course $f(x) = f(x)$, which implies $(x, x) \in R$.

R is symmetric because if $(x, y) \in R$, then $f(x) = f(y)$ which implies $f(y) = f(x)$, and hence $(y, x) \in R$.

R is transitive because if $(x, y) \in R$ and $(y, z) \in R$, then $f(x) = f(y) = f(z)$, proving $(x, z) \in R$.

(B). First we prove $\#A \leq \#B$. Write $a_1, \ldots, a_n$ to denote the elements of A. We know that $f(a_j) \in B$ for each $j = 1, 2, \ldots n$, and that $f(a_j) \neq f(a_k)$ unless $a_j = a_k$, since f is one-to-one. Hence B contains at least n distinct elements, namely $f(a_1), f(a_2), \ldots, f(a_n)$. This implies $\#B \geq \#A$.

Now we show $\#A \geq \#B$. Write $b_1, \ldots, b_m$ to denote the elements of B. Since f is onto, we know there are $x_1, x_2, \ldots, x_m \in A$ such that $f(x_j) = b_j$ for each $j \in \{1, \ldots m\}$. Since f is a function, we know the elements $x_1, \ldots, x_m$ are distinct– in other words, A contains at least m elements, proving that $\#A \geq \#B$.

Since $\#A \leq \#B$ and $\#A \geq \#B$, we know $\#A = \#B$.

5. **(10 points) Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $Y = \{1, 2, 3, 4, 5\}$.**
(A) (5 points) How many distinct functions are there from X to Y?
(B) (5 points) How many distinct one-to-one functions are there from X to Y?

(A) There are 5 possibilities for $f(1)$, 5 possibilities for $f(2)$, $\ldots$, and 5 possibilities for $f(10)$. Hence there are 5^{10} possibilities for f.

(B) There are no one-to-one functions from X to Y. We can see this by looking at the proof in the last problem that $\#A \leq \#B$. In this problem, that would imply $10 = \#X \leq \#Y = 5$, which is false. Hence there can be no one-to-one functions from X to Y. Alternatively, we could use the multiplication principle: There are 5 possibilities for $f(1)$, 4 possibilities for $f(2)$, 3 possibilities for $f(3)$, 2 possibilities for $f(4)$, 1 possibility for $f(5)$, zero possibilities for $f(6)$, etc. Multiplying gives us $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 = 0$.

6. **(10 points) Let $n \geq 1$ be an integer.**
(A) Prove

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

by using the binomial theorem. (3 points) (Remember that $\sum_{k=0}^{n} C(n,k) = C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n)$.)
(B) Prove the same statement WITHOUT using the binomial theorem. (7 points)

(A) Recall that the binomial theorem says

$$(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k} C(n,k).$$

Using $a = b = 1$ gives us $2^n = \sum_{k=0}^{n} C(n,k)$.

(B) We know that a set with n elements has 2^n total subsets. We also know that a set with n elements has $C(n,k)$ subsets of size k. Since the total number of subsets is the number of subsets of size 0 plus the number of subsets of size 1 plus the number of subsets of size 2 plus ... plus the number of subsets of size n, we have

$$2^n = C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n) = \sum_{k=0}^{n} C(n,k),$$

which is what we wanted to prove.

7. **(10 points) If $x_1, x_2, \ldots, x_n$ are real numbers in the interval $[0,1]$, prove that**

$$(1-x_1)(1-x_2)\cdots(1-x_n) \geq 1 - x_1 - x_2 - \cdots - x_n.$$

We prove this by induction. First, if $n = 1$, this is obvious, since of course $1-x_1 \geq 1-x_1$. Now assume that for some n, we know

$$\prod_{j=1}^{n}(1-x_j) \geq 1 - \sum_{j=1}^{n} x_j.$$

(This is the induction hypothesis.) We must now prove

$$\prod_{j=1}^{n+1}(1-x_j) \geq 1 - \sum_{j=1}^{n+1} x_j.$$

Note that by the induction hypothesis,

$$\begin{aligned}
\prod_{j=1}^{n+1}(1-x_j) &= (1-x_{n+1})\prod_{j=1}^{n}(1-x_j) && (1)\\
&\geq (1-x_{n+1})\left(1-\sum_{j=1}^{n} x_j\right) && (2)\\
&= 1-\sum_{j=1}^{n} x_j - x_{n+1} + x_{n+1}\left(\sum_{j=1}^{n} x_j\right) && (3)\\
&\geq 1-\sum_{j=1}^{n} x_j - x_{n+1} && (4)\\
&= 1-\sum_{j=1}^{n+1} x_j, && (5)
\end{aligned}$$

which is what we wanted to prove. Notice that the expression in line (3) is bigger than the expression in line (4) because the last term in (3) is always nonnegative.