# Midterm 2

# Math 61

Fall, 2016

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Section: 1B

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

| | |
|---|---|
| 1 | 15 |
| 2 | 13 |
| 3 | ~~15~~ 14 |
| 4 | 5 |
| 5 | 10 |
| Total | 57 |

10

1. (a) (10 pts) Suppose $G = (V, E)$ is a simple graph with $n$ vertices. Prove that if $\deg(v) \geq \frac{n-1}{2}$ for every vertex $v \in V$, then $G$ is connected.

Suppose that G has 2 components.

Since each vertex in either component has deg $\geq \frac{n-1}{2}$,

there must be $\frac{n-1}{2} + 1$ vertices at least in each component

$= \frac{n+1}{2}$

If we have at least two components, there must be $\frac{n+1}{2}$ vertices at least in each component.

$\Rightarrow$ Total no. of vertices $= \frac{n+1}{2} + \frac{n+1}{2} = n+1$ vertices.

However, since we have only n vertices, the graph cannot have 2 or more components.

$\Rightarrow$ G is connected.

5

(b) (5 pts) Suppose $G = (V, E)$ is a simple graph with $n$ vertices where $n > 1$. Prove that there must be two different vertices of $G$ that have the same degree.

The possible numbers for the degree of each vertex is

$0 \ldots\ldots n-1$. (n of those)

However, if

1. a vertex has degree $(n-1)$, then it must be connected to every other vertex and no other vertex can have degree 0.

   $\Rightarrow (n-1)$ possible degrees

2. a vertex has degree (0), then no other vertex can have degree $(n-1)$.

   $\Rightarrow (n-1)$ possible degrees.

In either of the cases, we have n vertices and $(n-1)$ possible degrees.

Since $n > n-1$, using the Pigeonhole principle, there are at least two different vertices with the same degree.

3

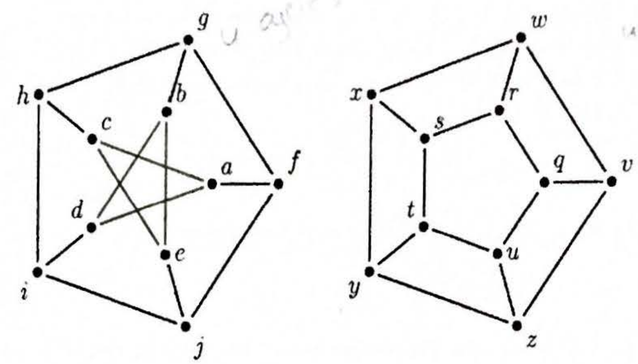
2. (a) (5 pts) State the definition of when two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic.

Two graphs $G_1, G_2$ are isomorphic if there is a bijection $f: V_1 \to V_2$ such that if $u, v \in V_1$ and u and v are connected, ~~then~~ iff $f(u), f(v)$ must be connected in $G_2$.

(b) (5 pts) Are the following two graphs isomorphic? If so, give an isomorphism using the table below. If not, explain why.

✓

$a \mapsto$
$b \mapsto$
$c \mapsto$
$d \mapsto$
$e \mapsto$
$f \mapsto$
$g \mapsto$
$h \mapsto$
$i \mapsto$
$j \mapsto$

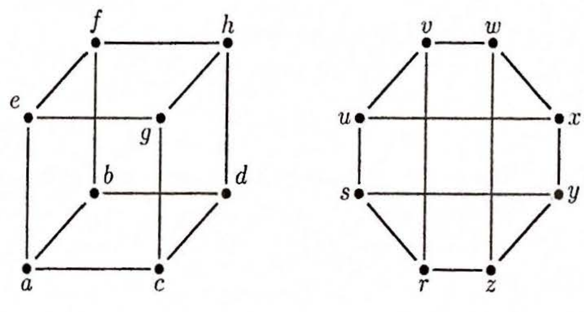



The graphs are not isomorphic
The first graph has no 4-cycles whereas the second graph has at least 5 - 4 cycles.

(c) (5 pts) Are the following two graphs isomorphic? If so, give an isomorphism using the table below. If not, explain why.

✓

$a \mapsto r$
$b \mapsto v$
$c \mapsto z$
$d \mapsto w$
$e \mapsto s$
$f \mapsto u$
$g \mapsto y$
$h \mapsto x$




The graphs are isomorphic.



13

14

3. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: If $G$ is a graph with $n$ vertices, then any path of length $n$ in $G$ must include some vertex at least twice.

T / F: Suppose $A$ is the adjacency matrix of a graph $G$ with $n$ vertices. If $G$ is connected, then every entry of the matrix $A^n$ is nonzero.

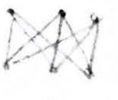
T / F: Suppose $G$ is a graph with weight function $w$, fix a vertex $a$, and for every vertex $v$ in $G$, let $L(v)$ be the length of the shortest path from $a$ to $V$. If we use Dijkstra's algorithm to find $L(v)$, then before algorithm returns $L(v)$, the algorithm first correctly finds $L(u)$ for every vertex $u$ such that $L(u) < L(v)$.

T / F: There are $(n-1)!$ different isomorphisms from $K_n$ to $K_n$.

T / F: If $G$ is a simple graph that has an Euler cycle, and $G'$ is a subgraph obtained from $G$ by removing only a single edge and removing no vertices, then $G'$ cannot have an Euler cycle.

T / F: There is a simple graph with 6 vertices whose vertices have degree 0, 1, 2, 3, 4, and 4.

T / F: The graph $K_{3,3}$ has exactly $3^2 2^4 = 144$ simple cycles.



T / F: $4^n = \sum_{i=0}^{n} 2^n C(n,i)$ for every integer $n \geq 0$.

$(2+2)^n = \sum_{i=0}^{n} C(n,i) 2^i 2^{n-i}$

T / F: The general solution for the recurrence $a_n = 6a_{n-1} + 9a_{n-2} + 2^n$ is $a_n = b_1 3^n + b_2 2^n$ where $b_1, b_2$ are constants.

T / F: If $P(n,k)$ is the number of partitions of $\{1, \ldots, n\}$ into $k$ many pieces, then for $n, k > 1$,

$$P(n,k) = \sum_{i=1}^{k} C(n,i) P(n-i, k-1).$$

$x^2 - 6x - 9 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(1)(-9)}}{2} = \frac{6 \pm \sqrt{72}}{2} = \frac{6 \pm 6\sqrt{2}}{2} = 3 \pm 3\sqrt{2} = 3+3\sqrt{2}$ or $3 - 3\sqrt{2}$

$C_0 2^n = 6C_0 2^{n-1} + 9C_0 2^{n-2} + 2^n$

$4C_0 = 12C_0 + 9C_0 + 2^n$

$b_1(3+3\sqrt{2})^n + b_2(3-3\sqrt{2})^n$

4. (a) (5 pts) Show that $C(i,k) = C(i+1,k+1) - C(i,k+1)$ for $i > k$.

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$$C(i,k) = \frac{i!}{k!(i-k)!}$$

$$C(i+1,k+1) - C(i,k+1) = \frac{(i+1)!}{(k+1)!(i-k)!} - \frac{i!}{(k+1)!(i-k-1)!} = \frac{i!}{(k+1)!(i-k-1)!}\left[\frac{(i+1)}{(i-k)} - 1\right]$$

$$= \frac{i!}{(k+1)!(i-k-1)!}\left[\frac{\cancel{i}+1-\cancel{i}+k}{(i-k)}\right] = \frac{i!\,\cancel{(k+1)}}{\cancel{(k+1)}(k!)(i-k)(i-k-1)!}$$

$$= \frac{i!}{k!(i-k)!} = C(i,k)$$

Hence, proved.

(b) (5 pts) Show that

$$C(n+1,k+1) = \sum_{i=k}^{n} C(i,k).$$

0

$$C(n+1,k+1) = \frac{(n+1)!}{(k+1)!(n-k)!}$$

$$\sum_{i=k}^{n} C(i,k) = C(k,k) + C(k+1,k) + C(k+2,k) + \ldots\ldots + C(n-1,k) + C(n,k)$$

$$= C(k,0) + C(k+1,1) + C(k+2,2) + \ldots\ldots + C(n-1,n-k-1) + C(n,n-k)$$

16

5. (a) (5 pts) Find the general solution for the recurrence $a_n = -4a_{n-1} - 4a_{n-2}$.

Our characteristic polynomial is

$x^2 + 4x + 4 = 0$ $\Rightarrow (x+2)^2 = 0$

$\Rightarrow x = -2$

Since we have only one soln to the characteristic polynomial,

$a_n = b_1(-2)^n + b_2 n(-2)^n$

where $b_1, b_2$ are constants.

(b) (5 pts) Find the solution to the recurrence $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 3$ and $a_1 = 0$.

Cont. from first part

$a_0 = b_1 = 3$ $\Rightarrow b_1 = 3$

$a_1 = -2b_1 - 2b_2 = 0$ $\Rightarrow 2b_2 = -2b_1$

$\Rightarrow b_2 = -b_1$

$\Rightarrow b_2 = -3$

$\Rightarrow$ Solution to the recurrence is

$a_n = 3(-2)^n - 3n(-2)^n$