Math 61

Midterm 2

Fall, 2016

Name: Key

SID:

Section:

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

1	
2	
3	
4	
5	
Total	

1. (a) (5 pts) Show that $C(i,k) = C(i+1,k+1) - C(i,k+1)$ for $i > k$.

Rewriting, $C(i+1,k+1) = C(i,k+1) + C(i,k)$
the LHS is the number of ways to choose $k+1$ numbers from $\{1,\ldots,i+1\}$. Counting a different way, there are $C(i,k+1)$ many ways to choose $k+1$ numbers from $\{1,\ldots,i\}$, and $C(i,k)$ many ways to choose $k+1$ numbers from $\{1,\ldots,i+1\}$ if we must include $i+1$ (since there are k other numbers to choose from $\{1,\ldots,i\}$). So the RHS counts the same collection.

One can also do this problem just using algebra.

(b) (5 pts) Show that

$$C(n+1,k+1) = \sum_{i=k}^{n} C(i,k).$$

The LHS is the number of ways of choosing $k+1$ numbers from the set $\{1,\ldots,n+1\}$. We show the RHS also counts the number of ways of doing this. If we choose $k+1$ numbers from $\{1,\ldots,n+1\}$, the largest number must be between $k+1$ and $n+1$. If we let $i+1$ be this largest number, the remaining k numbers must come from $\{1,\ldots,i\}$. So the number of ways to choose the largest number first, and then the remaing k numbers is $\sum_{i=k}^{n} C(i,k)$.

One can also do this problem just using part a and canceling adjacent terms in the summation.

2. (a) (5 pts) Find the general solution for the recurrence $a_n = -4a_{n-1} - 4a_{n-2}$.

The characteristic polynomial is $x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

so the only root is $x = -2$

The general solution is

$a_n = b_1 (-2)^n + b_2 n(-2)^n$

where b_1, b_2 are constants.

(b) (5 pts) Find the solution to the recurrence $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 3$ and $a_1 = 0$.

Solving to find b_1 and b_2,

$a_0 = 3 = b_1$

$a_1 = 0 = -2b_1 - 2b_2$, so $b_2 = -3$

So $a_n = 3(-2)^n - 3n(-2)^n$

3. (a) (10 pts) Suppose $G = (V, E)$ is a simple graph with n vertices. Prove that if $\deg(v) \geq \frac{n-1}{2}$ for every vertex $v \in V$, then G is connected.

Let u and v be any two vertices. We must show there is a path from u to v. We may as well assume that $u \neq v$ since there is a path (of length 0) from u to u. If there is an edge from u to v then we are done. So assume there is no edge from u to v. We claim that there is a vertex w so that u is adjacent to w and v is adjacent to w. This is true because if all the vertices adjacent to u and v were different, since $\deg(u) \geq \frac{n-1}{2}$ and $\deg(v) \geq \frac{n-1}{2}$, this would be at least $n-1$ vertices. But there are only $n-2$ vertices not equal to u or v, since the graph has n vertices.

(b) (5 pts) Suppose $G = (V, E)$ is a simple graph with n vertices where $n > 1$. Prove that there must be two different vertices of G that have the same degree.

Case 1: $\deg(v) \geq 1$ for every $v \in V$. Then by the pigeonhole principle, since there are n vertices, and the degree of each vertex must be between 1 and $n-1$, two vertices have the same degree.

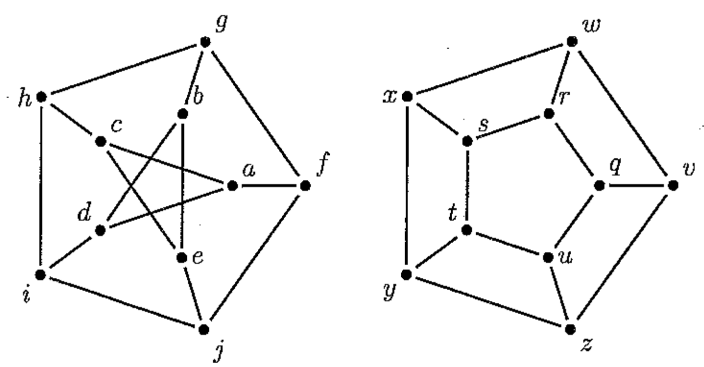
Case 2: $\deg(v) = 0$ for some $v \in V$. Then there cannot be any vertex with degree $n-1$, since it would have to have an edge to every other vertex, including v. So there are n vertices, and each vertex must have degree between 0 and $n-2$. So once again, by the pigeonhole principle, there must be two vertices that have the same degree

4. (a) (5 pts) State the definition of when two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic.

If there exists a bijection $f: V_1 \to V_2$ such that for all $u, v \in V_1$ there is an edge from u to v in G_1 iff there is an edge from $f(u)$ to $f(v)$ in G_2.

(b) (5 pts) Are the following two graphs isomorphic? If so, give an isomorphism using the table below. If not, explain why.

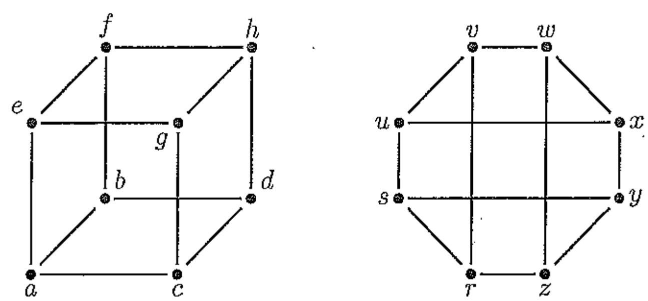
$a \mapsto$
$b \mapsto$
$c \mapsto$
$d \mapsto$
$e \mapsto$
$f \mapsto$
$g \mapsto$
$h \mapsto$
$i \mapsto$
$j \mapsto$



No. The graph on the left has no ~~4-cycles~~ simple cycles of length 4. The graph on the right has simple cycles of length 4. For example, t, u, z, y, t.

(c) (5 pts) Are the following two graphs isomorphic? If so, give an isomorphism using the table below. If not, explain why.

$a \mapsto r$
$b \mapsto s$
$c \mapsto z$
$d \mapsto y$
$e \mapsto v$
$f \mapsto u$
$g \mapsto w$
$h \mapsto x$



5. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: There is a simple graph with 6 vertices whose vertices have degree 0, 1, 2, 3, 4, and 4.

T / F: The graph $K_{3,3}$ has exactly $3^2 2^4 = 144$ simple cycles.

T / F: $4^n = \sum_{i=0}^{n} 2^n C(n,i)$ for every integer $n \geq 0$.

T / F: The general solution for the recurrence $a_n = 6a_{n-1} + 9a_{n-1} + 2^n$ is $a_n = b_1 3^n + b_2 2^n$ where b_1, b_2 are constants.

T / F: If $P(n,k)$ is the number of partitions of $\{1, \ldots, n\}$ into k many pieces, then for $n, k > 1$,

$$P(n,k) = \sum_{i=1}^{k} C(n,i) P(n-i, k-1).$$

T / F: If G is a graph with n vertices, then any path of length n in G must include some vertex at least twice.

T / F: Suppose A is the adjacency matrix of a graph G with n vertices. If G is connected, then every entry of the matrix A^n is nonzero.

T / F: Suppose G is a graph with weight function w, fix a vertex a, and for every vertex v in G, let $L(v)$ be the length of the shortest path from a to V. If we use Dijkstra's algorithm to find $L(v)$, then before algorithm returns $L(v)$, the algorithm first correctly finds $L(u)$ for every vertex u such that $L(u) < L(v)$.

T / F: There are $(n-1)!$ different isomorphisms from K_n to K_n.

T / F: If G is a simple graph that has an Euler cycle, and G' is a subgraph obtained from G by removing only a single edge and removing no vertices, then G' cannot have an Euler cycle.