Midterm 1

Math 61

Fall, 2016

Name: [redacted]

SID:

Section: A

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

1	12
2	9
3	6
4	11
5	15
Total	53

1. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: If X is a finite set with $|X| = k$, then there are k^n many strings over X of length n.

k k k k k

T / F: The number of onto functions from $\{1, 2, 3, 4, 5\}$ to $\{1, 2\}$ is equal to $\frac{5!}{3!2!}2^3$.

2^3. $C_{5\,2}$

2 2 2 2 1 — 1 2 3 4 5 — 1 2

T / F: If A and B are finite sets and $|A \cup B| = |B|$, then $A \subseteq B$.

T / F: If f is a bijection from X to Y, then f^{-1} is a bijection from Y to X.

onto, one-one

1 — 2, 2 — 3, 3 — 4, 5 — 5

T / F: There are $\frac{(n+m)!}{n!m!}$ ways to divide n many identical balls into m many distinct boxes.

$* * * \mid * * \mid * * \mid * = C(n+m, m)$ $\quad \frac{(n+m)!}{n!m!}$

T / F: Consider the set S of strings of length 10 containing exactly four a's, three b's, and three c's. There are more strings in S ending with a than there are strings in S ending with b.

3a, 3bs, 3 cs vs four as, 2bs, 3cs

T / F: If $n \geq 3$, then there are $(n-1)!/2$ ways to seat n different people around a circular table where two seatings are considered identical if each person has the same (unordered) set of two neighbors.

$n = 4 = 3!$ 3ppl

T / F: The relation $\{(1, 2), (3, 1), (2, 3), (1, 1), (2, 2), (3, 3)\}$ on the set $\{1, 2, 3\}$ is transitive.

T / F: Suppose f is a function from X to Y. Then f is one-to-one if for every $x \in X$ there is a unique $y \in Y$ so that $f(x) = y$.

T / F: Suppose R is a relation on a set X and R is transitive and symmetric. Then for all $x, y \in X$, if $(x, y) \in R$, then $(x, x) \in R$.

If transitive and symmetric

If (x, y) then (y, x)

If (x, y) and (y, x), then (x, x)

12

2. (15 pts) Consider a standard deck of 52 playing cards, where each card is one of four different suits $\diamondsuit, \heartsuit, \clubsuit, \spadesuit$ and one of 13 different ranks: $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$, and all combinations of suits and ranks are possible.

A hand means 5 different cards where order does not matter.

For full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n,r)$ or $C(n,r)$.

(a) (5 pts) How many hands contain only cards whose rank is J or Q or K?

$4+4+4 = 12$ cards

$= C(12,5)$ (combination - choose a hand of 5 out of 12 cards)

5

$= \boxed{\dfrac{12!}{5!\,7!}}$

(b) (5 pts) How many hands are "three of a kind" (contain three cards of one rank, and the remaining cards have two other ranks)? solved very quickly

$= 13 \cdot C(4,3) \cdot C(48,1) \cdot C(44,1)$

combinations - order does not matter - pick 3 of 1 kind, multiply by 13 bc 13 possible cards, pick last two cards out of 44

2

$= 13 \cdot \dfrac{4!}{3!} \cdot \dfrac{48!}{2 \cdot 46!} = \boxed{13 \cdot 4 \cdot \dfrac{48!}{47!} \cdot \dfrac{44!}{43!}}$

(c) (5 pts) How many hands are "two pair" (contain two cards of the same rank, two cards of another rank, and one card of a third rank)?

$13 \cdot C(4,2) \cdot 12 \cdot C(4,2) \cdot C(46,1)$

2

$= \boxed{13 \cdot \dfrac{4!}{2! \cdot 2!} \cdot 12 \cdot \dfrac{4!}{2! \cdot 2!} \cdot 46}$

combinations,

3. **(10 pts) In the following problem, for full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n,r)$ or $C(n,r)$.**

(a) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_8 = 20$, where $x_1, x_2, \ldots, x_8$ are integers greater than or equal to 0?

$x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | x_8$ → stars and bars such that there are 20 stars and 7 bars =

$$\boxed{\frac{27!}{20!\,7!}}$$

(b) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_8 = 20$, where $x_1, x_2, \ldots, x_8$ are integers greater than or equal to 0, and least one of the variables $x_1, \ldots, x_8$ is equal to 0 or exactly one of the variables $x_1, \ldots, x_8$ is equal to 1?

1 variable = 0 → Pretend that doesn't exist (fixed)

= stars & bars with 6 bars, 20 stars =

$\frac{26!}{20!\,6!}$

exactly 1 variable = 1 : fix 1 star as 1, remove cases where other stars are 1 = 19 stars, 6 bars = $\frac{25!}{19!\,6!} - 7 \cdot \frac{23!}{18!\,5!}$

7 removed variables, 18 stars, 5 bars

both are true! 1 variable = 0, one = 1

= 5 bars, 19 stars - cases that there are 1s =

$\frac{24!}{19!\,5!} - 6 \cdot \frac{22!}{18!\,4!}$ ← 4 bars, 18 stars

So all together =

$$\boxed{\frac{26!}{20!\,6!} + \frac{25!}{19!\,6!} - 7 \cdot \frac{23!}{18!\,5!} - \frac{24!}{19!\,5!} + 6 \cdot \frac{22!}{18!\,4!}}$$

6

4. (15 pts)

Prove the following by induction for every $n \geq 1$. For all finite sets X and Y with $|X| = |Y| = n$, if f is a function from X to Y that is one-to-one, then f is onto.

3

(a) (3 pts) State and prove the base case.

$n=1$: $|X| = |Y| = 1$, then there is 1 element in X and Y. If the function is one to one, the only possible one to one function is $x \in X$ to $y \in Y$ where x and y are the only elements in the set, which is onto (everything in Y is used up). ✓

8

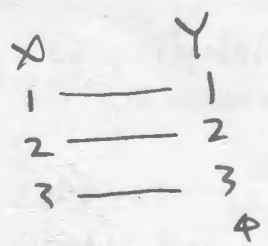
(b) (9 pts) Prove the inductive step.

for $|X| = |Y| = n+1$ – If we remove 1 element x from X and one element y from Y, we get $|X'| = |Y'| = n$ which is one to one and onto by induction. what is onto and 1-1? (the restriction of the fcn)

There fore if we add in x and y, the only possible additional connection is $x \to y$ because the function must be one to one so all other elements in Y must already be "used". Since y is also now connected to, the function is also onto ✓.

(c) (3 pts) Either prove the following or give a counterexample: For every set X, if f is a one-to-one function from X to X, then f is onto.

0



Not onto

$\boxed{\text{False}}$

15

5. (15 pts)

(a) (5 pts) Write the definition of what it means for a relation R on a set X to be an equivalence relation.

R is symmetric, transitive, and reflexive.

ordered pair of elements (x,y) in set X.

(b) (6 pts) Let E be the relation on the set of all positive real numbers $\mathbb{R}^+$ where $x\ E\ y$ if $x/y = 2^n$ for some integer $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. For example, $(2/3)\ E\ (8/3)$, since $\frac{2/3}{8/3} = \frac{1}{4} = 2^{-2}$. Show that E is an equivalence relation.

symmetric: If xRy then $\frac{x}{y} = 2^n \rightarrow x = 2^n y \rightarrow 1 = 2^n \frac{y}{x} \rightarrow \frac{y}{x} = \frac{1}{2^n} \rightarrow \frac{y}{x} = 2^{-n}$ so yRx because $-n$ is always an integer.

reflexive: xRx so $\frac{x}{x} = 2^n \rightarrow 1 = 2^n \rightarrow 1 = 2^0 \rightarrow 1 = 1$ ✓ always true for $n=0$.

transitive: If xRy and yRz then xRz so $\frac{x}{y} = 2^n$ and $\frac{y}{z} = 2^k \rightarrow y = z2^k \rightarrow \frac{x}{z2^k} = 2^n \rightarrow \frac{x}{z} = 2^k 2^n \rightarrow \frac{x}{z} = 2^{k+n}$ where k and n are integers, so xRz ✓

(c) (4 pts) What is the equivalence class of 1 with respect to the relation E?

$[1] = \{y \in \mathbb{R} : \frac{1}{y} = 2^n\}$ → so this is true for all integers n such that $y = 2^{-n}$

ex: $(\ldots, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots)$

so solve

$\frac{1}{y} = 2^n$

$1 = 2^n y$

$\frac{1}{2^n} = y$

$2^{-n} = y$