Math 61

Midterm 1

Fall, 2016

Name:

SID:

Section:

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

1	15
2	10
3	5
4	15
5	18
Total	63

15

1. (15 pts)

(a) (5 pts) Write the definition of what it means for a relation R on a set X to be an equivalence relation.

A relation is an equivalence relation if it is

reflexive

symmetric

and transitive.

(b) (6 pts) Let E be the relation on the set of all positive real numbers $\mathbb{R}^+$ where $x\ E\ y$ if $x/y = 2^n$ for some integer $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. For example, $(2/3)\ E\ (8/3)$, since $\frac{2/3}{8/3} = \frac{1}{4} = 2^{-2}$. Show that E is an equivalence relation.

Reflexive if $\forall x \in \mathbb{R}^+$, $x E x$

$\frac{x}{x} = 1 = 2^0$, so E is reflexive

Symmetric if $\forall x, y \in \mathbb{R}^+$, if xEy then yEx

if $\frac{x}{y} = 2^n$, then $\frac{y}{x} = 2^{-n}$ $-n$ is an integer, so E is symmetric

Transive: if $\forall x,y,z \in \mathbb{R}^+$, if xEy and yEz, then xEz

let $\frac{x}{y} = 2^n$ $\frac{y}{z} = 2^m$

then $\frac{x}{y} \cdot \frac{y}{z} = 2^n \cdot 2^m = 2^{n+m}$. $(n+m)$ is an integer, so E is transitive

(c) (4 pts) What is the equivalence class of 1 with respect to the relation E?

$[x]_E$ is the set of all $y \in X$ s.t. $(x,y) \in E$

$[x]_E = \{ y \in \mathbb{R}^+ \mid y = 2^{-n}$ where n is an integer$\}$

(or the set of powers of 2)

$\frac{1}{y} = 2^n$

$y = \frac{1}{2^n} = 2^n$

2. (15 pts)

Prove the following by induction for every $n \geq 1$. For all finite sets X and Y with $|X| = |Y| = n$, if f is a function from X to Y that is one-to-one, then f is onto.

(a) (3 pts) State and prove the base case.

let $X = \{x\}$
$Y = \{y\}$

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For $n=1$, both $|X| = |Y| = 1$

Since f is one-to-one, $\forall x \in X$ there is exactly one $y \in Y$ such that $f(x) = y$.

This means that the one element in each set are mapped to eachother.

f is onto because $\forall y \in Y$, there is at least one x s.t $f(x) = y$ because $f(x) = y$. ✓

(b) (9 pts) Prove the inductive step.

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Suppose for $n = k$, $|X| = |Y| = k$ if $f: X \to Y$ is one-to-one, the f is onto.

Show: for $n = k+1$, f is onto if $f: X \to Y$ is one-to-one.

When $|X| = |Y| = k$, every element in X must map to exactly one element in Y, and vice versa. This is because $\forall x_1, x_2 \in X$, when $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ (def of 1-to-1). And if every element in Y must have a corresponding x s.t $f(x) = y$.

By adding one element to sets X and Y so that $|X| = |Y| = k+1$, we know that the ~~last elements must map to eachother~~ since f is 1-to-1 ~~and onto~~. The previous elements are onto b/c of our hypothesis.

↳ this means it is onto.

not true for all 1-1 functions

(c) (3 pts) Either prove the following or give a counterexample: For every set X, if f is a one-to-one function from X to X, then f is onto.

False.

Let
$X = \{1, 2\}$
$Y = \{1, 2, 3\}$

$f := \{(1,1)\ (2,2)\}$

f is one-to-one but not onto.

0

3. (10 pts) **In the following problem, for full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n,r)$ or $C(n,r)$.**

(a) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_9 = 20$, where $x_1, x_2, \ldots, x_9$ are integers greater than or equal to 0?

* * | * * | * * * | * || * * * * * | * * | * * | * * *

2 2 3 1 0 5 2 2 3

x_1 x_2 - - - x_9

Arrangement of strings with 20 *'s and 8 bars.

The bars will separate the stars and will give sections that correspond to the values of $x_1 - x_9$.

There are $C(28,8)$ such strings. (choose 8 places to puts bars in the string)

$$C(28,8) = \frac{28!}{(28-8)!\,8!} = \frac{28!}{20!\,8!}$$

(b) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_9 = 20$, where $x_1, x_2, \ldots, x_9$ are integers greater than or equal to 0, and least one of the variables $x_1, \ldots, x_9$ is equal to 0 or exactly one of the variables $x_1, \ldots, x_9$ is equal to 1?

0

$|X \cup Y| = |X| + |Y| - |X \cap Y|$

Either: at least one $x_i = 0$

OR: exactly one $x_i = 1$

Count complement

A = # of strings of all $x_i \geq 1$

B = # strings all ≥ 1, & exactly one $x_i = 1$

Claim: Answer = Total # $- |A - B|$

$|A-B|$ = all strings where all $x_i \geq 1$, not exactly $x_i = 1$

$|A|$ = $20 - 9$ stars $= 11$ 8 bars $\rightarrow C(19,8)$

$|B|$: choose the $x_i = 1$. The remaining $x_i \geq 2$

$\underset{\text{remain}}{19} - \underset{\text{8 buckets}}{8}\,(\underset{x_i \geq 2}{2}) = 19 - 16 = 3$ stars, 7 bars ($x_i = 1$)

$\hookrightarrow C(10,3)$

Ans: $\binom{28}{8} - |A-B|$

$= \binom{28}{8} - \binom{19}{8} + \binom{10}{3}$

5

4. (15 pts) Consider a standard deck of 52 playing cards, where each card is one of four different suits $\diamondsuit, \heartsuit, \clubsuit, \spadesuit$ and one of 13 different ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and all combinations of suits and ranks are possible.

A hand means 5 different cards where order does not matter.

For full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n, r)$ or $C(n, r)$.

(a) (5 pts) How many hands contain only cards whose rank is J or Q or K?

\# of cards $J, Q, K = 4 \cdot 3 = 12$

choose 5 cards:

5

$$\binom{12}{5} = \frac{12!}{(12-5)!\,5!} = \frac{12!}{7!\,5!}$$

(b) (5 pts) How many hands are "three of a kind" (contain three cards of one rank, and the remaining cards have two other ranks)?

5

$$\underbrace{\binom{13}{1}}_{\text{choose rank for triple}} \underbrace{\binom{4}{3}}_{\text{choose 3 cards from rank}} \cdot \underbrace{\binom{12}{2}}_{\text{choose 2 ranks from remaining 12}} \underbrace{\binom{4}{1}\binom{4}{1}}_{\text{choose 1 card from each rank}}$$

$$= \frac{13!}{12!\,1!} \cdot \frac{4!}{3!\,1!} \cdot \frac{12!}{10!\,2!} \cdot \frac{4!}{3!\,1!} \cdot \frac{4!}{3!\,1!} = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4$$

(c) (5 pts) How many hands are "two pair" (contain two cards of the same rank, two cards of another rank, and one card of a third rank)?

5

$$\underbrace{\binom{13}{2}}_{\text{2 ranks for pairs}} \underbrace{\binom{4}{2}\binom{4}{2}}_{\text{pairs}} \cdot \underbrace{\binom{11}{1}}_{\text{last rank}} \underbrace{\binom{4}{1}}_{\text{last card}}$$

$$= \frac{13!}{11!\,2!} \cdot \frac{4!}{2!\,2!} \cdot \frac{4!}{2!\,2!} \cdot \cancel{\frac{11!}{10!\,1!}}^{\,11} \cdot \cancel{\frac{4!}{3!\,1!}}^{\,4}$$

$$= \frac{13!}{11!\,2!} \cdot 6 \cdot 6 \cdot 11 \cdot 4$$

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5. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: If A and B are finite sets and $|A \cup B| = |B|$, then $A \subseteq B$.

T / F: If X is a finite set with $|X| = k$, then there are k^n many strings over X of length n.

$\underbrace{k \cdot k}_{n\text{th}} = k^n$

T / F: If f is a bijection from X to Y, then f^{-1} is a bijection from Y to X.

T / F: There are $\frac{(n+m)!}{n!m!}$ ways to divide $\overset{3}{n}$ many identical balls into $\overset{5}{m}$ many distinct boxes. $\frac{(3+4)!}{3!\,4!} = \frac{7!}{3!\,4!}$

$x * | * | *$ $\frac{(n+m-1)!}{n!\,(n-1)!}$ $m-1$ bars

T / F: Consider the set S of strings of length 10 containing exactly four a's, three b's, and three c's. There are more strings in S ending with a than there are strings in S ending with b.

a: $\frac{9!}{3!\,3!\,3!}$ b: $\frac{9!}{4!\,2!\,3!}$ aaaabbbccc

more 32 32 2

T / F: Suppose f is a function from X to Y. Then f is one-to-one if for every $x \in X$ there is a unique $y \in Y$ so that $f(x) = y$.

needs to be exactly one y s.t. $f(x) = y$

T / F: Suppose R is a relation on a set X and R is transitive and symmetric. Then for all $x, y \in X$, if $(x, y) \in R$, then $(x, x) \in R$.

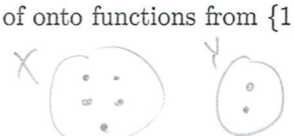
xRy yRx

xRy yRz xRz

T / F: The relation $\{(1, 2), (3, 1), (2, 3), (1, 1), (2, 2), (3, 3)\}$ on the set $\{1, 2, 3\}$ is transitive.

T / F: The number of onto functions from $\{1, 2, 3, 4, 5\}$ to $\{1, 2\}$ is equal to $\frac{5!}{3!2!}2^3$.

?



T / F: If $n \geq 3$, then there are $(n-1)!/2$ ways to seat n different people around a circular table where two seatings are considered identical if each person has the same (unordered) set of two neighbors.