Math 61

Midterm #1

Name SOLUTIONS

Bruin ID

Signature

TA/Section

You have 50 minutes.
There are 5 problems.
Make sure to show all work if you want to get full credit.
No notes, books, calculators, smartphones,... are allowed.

GOOD LUCK!

Question	Points	Your Score
Q1	9	
Q2	9	
Q3	9	
Q4	9	
Q5	14	
TOTAL	50	

Problem 1 (9 points) Show by induction that the following equation is true for every positive integer n

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Step 1: $n = 1$

$LHS = 1^2 = 1$

$RHS = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6} = 1$

so $LHS = RHS$ ✓

Step 2:

Suppose that

$$1^2 + 2^2 + \_ + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{holds.}$$

We show that

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \quad \text{is true.}$$

Using the inductive assumption,

$$LHS = 1^2 + 2^2 + \_ + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{(n+1)(2n^2 + n + 6n + 6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} =$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = RHS$$ ✓

Problem 2 (9 points) Find the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 25$$

satisfying $x_1 > 0, x_2 > 3, x_3 \geq 0, x_4 \geq 0$. Explain carefully your answer.

Let $x_1 = x_1' + 1$ and $x_2 = x_2' + 4$

Note that $x_1 > 0$ if and only if $x_1' \geq 0$ and

$x_2 > 3$ — // — $x_2' \geq 0$

We therefore, equivalently have to find the number of integer solutions to

$$(x_1' + 1) + (x_2' + 4) + x_3 + x_4 = 25,$$

$$\text{i.e. } x_1' + x_2' + x_3 + x_4 = 20,$$

where $x_1', x_2', x_3, x_4 \geq 0$

As we have seen in class, this number of solutions is equal to the number of choosing $4-1$ slots (this is where we put a bar |) from $20 + (4-1)$ slots.

The answer is $C(20 + (4-1), 4-1) = \boxed{C(23,3)}$

Problem 3 (9 points) Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{Z}$ be the set of integers. Define

$$R = \{(r, s) \in \mathbb{R} \times \mathbb{R} : r - s \in \mathbb{Z}\}.$$

a)(7 points) Show that R is an equivalence relation.

- R is reflexive : Let $r \in \mathbb{R}$. Then $r - r = 0 \in \mathbb{Z}$.
- R is symmetric : When $r, s \in \mathbb{R}$ and $r - s \in \mathbb{Z}$, then $s - r = -(r - s) \in \mathbb{Z}$
- R is transitive : Let $r, s, t \in \mathbb{R}$. Suppose that $r - s \in \mathbb{Z}$ and $s - t \in \mathbb{Z}$. Then $r - t = (r - s) + (s - t) \in \mathbb{Z}$.

b)(2 points) Compute the equivalence class of -3.7.

This is

$\ldots, -3.7, -2.7, -1.7, -0.7, 0.3, 1.3, 2.3, \ldots$

$= \{-3.7 + n : n \in \mathbb{Z}\}$

Problem 4 (9 points - 3 points each) You have a standard deck of 52 cards. Recall that there are 4 suits: $\heartsuit, \diamondsuit, \clubsuit, \spadesuit$, and there are 13 denominations: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2. In questions below, explain your answers.

a) In how many ways you can select 5 cards so that there are in the same suit?

$$\underbrace{4}_{\text{number of different suits}} \cdot C(\underbrace{13}_{\text{number of cards in a suite}}, 5)$$

b) In how many ways you can select 5 cards so that 2 of them are in one suit and 3 are in a second suit?

$$\underbrace{4}_{\text{pick 1st suit}} \cdot \overbrace{C(13,2)}^{\text{choose 2 cards from the suit}} \cdot \underbrace{3}_{\text{pick 2nd suit}} \cdot \overbrace{C(13,3)}^{\text{choose 3 cards from the suit}}$$

c) In how many ways you can select 5 cards so that 2 of them are in one suit, 2 are in a second suit, and 1 is in a third suit?

$$\underbrace{C(4,2)}_{\text{pick 2 suits}} \cdot \overbrace{C(13,2) \cdot C(13,2)}^{\text{pick 2 cards in each of these suits}} \cdot \underbrace{2}_{\text{pick 3rd suit}} \cdot \overbrace{C(13,1)}^{\text{pick the last card}}$$

Problem 5 (14 points - 2 points each) Each of the statements below is either true or false. Circle the correct answer. No justification is needed.

(a) ~~The~~ relation $R = \{(1,2),(1,1),(3,3),(3,2),(2,2)\}$ on the set $\{1,2,3\}$ is reflexive.
True **False**

(b) The relation $R = \{(1,2),(1,1),(3,3),(3,2),(2,2)\}$ on the set $\{1,2,3,4\}$ is reflexive.
True **False**

Note : $(4,4) \notin R$

(c) Let $g\colon X \to Y$ and $f\colon Y \to Z$ be functions. If $f \circ g$ is onto, then f is onto.
True **False**

(d) In the expansion of $(a+b)^5$ the coefficient at a^2b^3 is equal to 5.
True **False**

It is 10

(e) The sequence $\{a_n\}_{n=0}^{\infty}$ given by $a_n = (-1)^n$ is increasing.
True **False**

(f) Ann, Ben, Tom, and Kathy are waiting in a line in Starbucks. Suppose that Tom is standing right behind Ben. There are 4! many different lines they can form.
True **False**

(g) Let R and S be relations on X. If R and S are antisymmetric, then $R \cap S$ is antisymmetric.
True **False**

Scratch paper