Math 61
Fall 2017
12/9/17
Time Limit: 180 Minutes

Name (Print): Weijia Shi
SID Number: 104757423

Day \ T.A.	Eric	David	Chris
Tuesday	2A	2D	2F
Thursday	2B	2C	2E

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, circle your section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

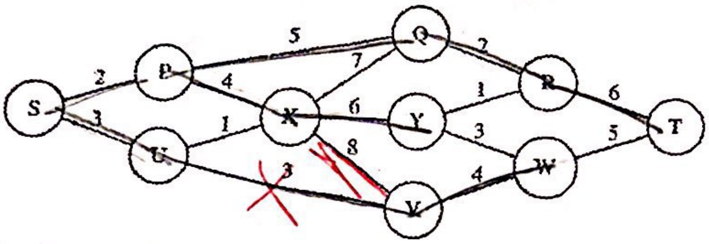
You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

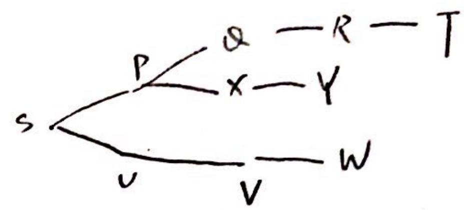
Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	5	4
2	5	5
3	5	5
4	5	3
5	5	4
6	5	5
Total:	30	26

1. (5 points) Use the breadth-first algorithm to construct a spanning tree for the following graph. (The ordering of the vertices is alphabetic.) Is the tree you obtain of minimal weight? (Here, the *weight* of a spanning tree is the sum of the weights of all its edges.)

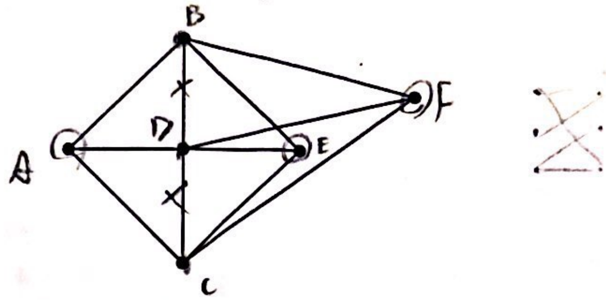


1. Starts at S
   - add edge SP
   - add edge SU
2. visit P
   - add PQ
   - add PX
3. visit U
   - add UV
4. visit Q
   - add QR
5. visit X
   - add XY
6. visit V
   - add VW
7. visit R
   - add RT

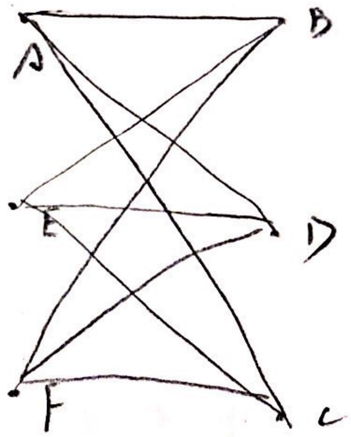


It's not minimal spanning tree because for edge incident on X the edge of minimal weight is not included in the tree

2. (5 points) Show that the following graph is not planar.



Redraw this graph
omit edge BD and DC



The graph has a subgraph isomorphic to $K_{3,3}$

3. (5 points) G is a tree with ~~6~~ 7 vertices, of which ~~3~~ 4 are leaves (=terminal vertices). What are the possible degrees of the ~~6~~ 7 vertices? (In particular, your answer should say why there are no other possibilities.) Draw an example of each.

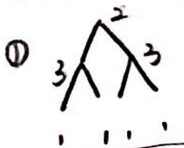
G rooted tree

Possible vertices

(1 1 1 1 2 3 3)

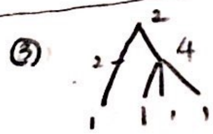
(1 1 1 1 1 2 5)

①



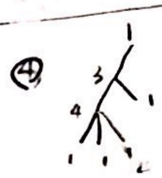
②

③



(1 1 1 1 2 2 4)

④



(1 1 1 1 1 3 4)

There are total 4 possibilities

The tree has $V-1=6$ edge

The total degree of the tree is $6\times2=12$

The total degree of leaves is 4

The total degree of interal vertices is 8

There are 3 internal vertices.

$x_1+x_2+x_3=8$

Suppose x_1 is the root, so $x_1 \geq 1$ let $y_1+1=x_1$

Suppose x_2, x_3 are internal vertices (not root)

so $x_2 \geq 2$, $x_3 \geq 2$ let $y_2+2=x_2$, $y_3+2=x_3$

The problem become. $y_1+1+y_2+2+y_3+2=8$

$y_1+y_2+y_3=3$

① if $y_1=0$, $y_2=1$, $y_3=2$
$x_1=1$, $x_2=3$ $x_3=4$

② if $y_1=1$, $y_2=1$, $y_3=1$
$x_1=2$, $y_2=3$. $y_3=3$

if $y_1=2$, $y_2=0$, $y_3=1$
$x_1=3$. $y_2=2$, $y_3=3$

same

③ if $y_1=3$, $y_2=0$, $y_3=0$
$x_1=4$, $y_2=2$, $y_3=2$

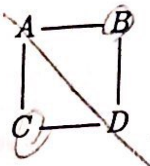
④ if $y_1=0$. $y_2=0$. $y_3=3$
$x_1=1$, $x_2=2$, $x_3=5$

There are total 4 possibilities

(situation: $y_1=2$. $y_2=0$. $y_3=1$
same as
$y_1=2$. $y_2=1$. $y_3=$

3

4. (5 points) A frog jumps on the vertices of a square, every time jumping to one of the two closest vertices. So, for example, from A it jumps to either B or C.



R_{n-1} R_{n-4} $2b_{n-1} = a_n$

In how many different ways can it get from A to A in n jumps?

Hint: it might be useful to consider the number of ways to get from A to B, C, and D in n jumps as well.

2
$$\begin{array}{c|c} & \begin{matrix} A & B & D \end{matrix} \\ \begin{matrix} A \\ B \\ D \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

3
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

4
$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

5
$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

6
$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{pmatrix}$$

We simplify the problem to just A, B, D 3 vertices because the graph is symmetric. The # path is 2 times the answer get from A, B, D graph.

We conclude # way from A to A is: if n is odd, $A \to A = 0$

if n is even, $A \to A = 2 \cdot 2^{[illegible]}$

5. (5 points) You don't need to simplify the formulas in this problem.

(a) Suppose you want to buy 12 items from an ample supply of glazed, chocolate, and powdered donuts. How many selections are possible?

(b) A 9-digit telephone number $d_1d_2d_3d_4d_5d_6d_7d_8d_9$ is called *memorable* if the sequence $d_1d_2d_3d_4$ is exactly the same as one of the sequences $d_5d_6d_7d_8$ or $d_6d_7d_8d_9$ (or both). Assume that each d_i can be any of the ten decimal digits 0,1,...,9. What is the number of memorable telephone numbers?

1234 4321

(c) In how many different orders can a spider put on its socks and shoes? (Of course, a spider has eight legs, and socks are put on before shoes.)

1 (a) $x_1 + x_2 + x_3 = 12$

$$C(14,2) = \frac{14\cdot 13}{2!} = 7\cdot 13$$

2 (b) $A: d_1d_2d_3d_4 = d_5d_6d_7d_8$

$B: d_1d_2d_3d_4 = d_6d_7d_8d_9$

$$|A\cup B| = |A| + |B| - |A\cap B|$$
$$= 10^4\cdot 10 + 10^4\cdot 10 - 10$$

1 (c) $8!\times 8!$.

2

2 2 2

$2!\cdot 2!$

4 4 2

$8!$ $8!$

$4\cdot 4\cdot 3$

$1+3$ 4 0

$3\cdot 3\cdot 2$

$3\cdot 4\cdot 2$

3· 1 2 3

6. (5 points) Prove that from any set of 1000 positive integers, one can choose either one number which is divisible by 1000, or several numbers whose sum is divisible by 1000.

Hint: Denote the numbers by $x_1, x_2, \ldots, x_{1000}$. Consider the sums

$$x_1, \quad x_1 + x_2, \quad \ldots, \quad x_1 + x_2 + \ldots + x_{1000}.$$

If one of them is divisible by 1000, then … Otherwise …

The remainder of 1000 is {0, 1, 2 ⋯ 999}

if the remainder is 0 ✓

Then we are done

if the remainder is not 0 1

The set of remainder {1, 2, 3 ⋯ 999} is the pidgenhole

the set of sum { x_1, x_1+x_2 ⋯, $x_1+x_2+\cdots+x_{1000}$ } is the pig

By PHP, 2 elements from the set of sum must have same remainder a

Suppose the 2 elements are $\sum_{i=1}^{j} x_i$, $\sum_{i=1}^{m} x_i$ ✓

$$\sum_{i=1}^{j} x_i \equiv \sum_{i=1}^{m} x_i \mod 1000.$$

So $\sum_{i=1}^{j} x_i = 1000k + a$

$$\sum_{i=1}^{m} x_i = 1000s + a$$

So $\sum_{i=1}^{j} x_i - \sum_{i=1}^{m} x_i = \sum_{i=m}^{j} x_i = 1000(k-s)$ if $j \geq m$

$$\sum_{i=1}^{r} x_i - \sum_{i=1}^{j} x_i = \sum_{i=j}^{m} x_i = 1000(s-k) \quad \text{if } m > j$$

Both $\sum_{i=m}^{j} x_i$ and $\sum_{i=j}^{m} x_i$ are divisible by 1000.

Scanned with CamScanner