Math 33B-3 Yeliussizov. Midterm 2

Exam time: 12:00-12:50 am, November 18, 2016

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Discussion section (NEITHALATH 3A Tue, 3B Thu; LEE 3C Tue, 3D Thu): 3A

There are 4 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

Problem 1 (20 pt)	Problem 2 (20 pt)	Problem 3 (20 pt)	Problem 4 (20 pt)	Total (80 pt)
20	20	2	15	57

Problem 1. (20 points) Find the solution to the following initial value problem

$$y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

$\lambda^2 - 6\lambda + 9 \to (\lambda - 3)^2 \to \lambda = 3$

$y_1 = e^{3t}, \quad y_2 = te^{3t}$

$y = C_1 e^{3t} + C_2 t e^{3t}$

$y' = 3C_1 e^{3t} + C_2(3te^{3t} + e^{3t})$

$= 3C_1 e^{3t} + C_2 3te^{3t} + C_2 e^{3t}$

$y(0) = 1 = C_1 + 0 \to C_1 = 1$

$y'(0) = 3 \cdot 1 \cdot 1 + C_2(0 + 1) = 3 + C_2 = 2 \to C_2 = -1$

$\boxed{y = e^{3t} - te^{3t}}$

$y' = 3e^{3t} - 3te^{3t} - e^{3t} = 2e^{3t} - 3te^{3t}$

$y'' = 6e^{3t} - 9te^{3t} - 3e^{3t} = 3e^{3t} - 9te^{3t}$

$\to 3e^{3t} - 9te^{3t} - \cancel{12e^{3t}} + \cancel{18t}e^{3t} + \cancel{9e^{3t}} - \cancel{9te^{3t}}$ ✓

20

$e^t - te^t \to \cancel{e^t} - te^t \cancel{e^t}$
$= -te^t$

Problem 2. (20 points) Determine the general solution to the following differential equation:

$$y'' + 3y' + 2y = xe^{-x}$$

$\lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) \to \lambda = -2, \lambda = -1$

$y_1 = e^{-2t}, \quad y_2 = e^{-t} \quad \to \quad y_p = v_1 y_1 + v_2 y_2$

$W = \begin{pmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{pmatrix} \quad \det(W) = e^{-2t} \cdot e^{-t} + 2e^{-2t} \cdot e^{-t} = e^{-3t} + 2e^{-3t} = e^{-3t}$

$W^{-1}g \quad e^{3t} \begin{pmatrix} -e^{-t} & -e^{-t} \\ 2e^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ te^{-t} \end{pmatrix} = \begin{pmatrix} -e^{2t} & -e^{2t} \\ 2e^{t} & e^{t} \end{pmatrix} \begin{pmatrix} 0 \\ te^{-t} \end{pmatrix}$

$= \begin{pmatrix} -te^{t} \\ t \end{pmatrix}$

$\frac{-e^{-t}(te^{-t})}{e^{-3t}} = -te^t$

$\frac{e^{-2t} te^{-t}}{e^{-3t}} = t$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \int \begin{pmatrix} -te^{t} \\ t \end{pmatrix} dt$

$v_1 = -\int te^t\,dt \qquad u = t,\ du = dt, \quad dv = e^t dt \to v = e^t$

$-(uv - \int v\,du) = -(te^t - \int e^t dt) = -te^t + e^t = e^t - te^t$

$v_2 = \int t\,dt = \frac{t^2}{2}$

$y_p = v_1 y_1 + v_2 y_2 = (e^t - te^t)e^{-2t} + \frac{t^2}{2}(e^{-t})$

$= e^{-t} - te^{-t} + \frac{t^2}{2}e^{-t} = e^{-t}\left(\frac{t^2}{2} - t + 1\right)$

$*$

$y = C_1 e^{-2t} + C_2 e^{-t} + e^{-t}\left(\frac{t^2}{2} - t + 1\right)$ ✓

20

Problem 3. (20 points) Consider the differential equation $x^2y'' - 3xy' + 4y = x^2$ for $x > 0$.
(a) Find the solution to the associated homogeneous equation $x^2y'' - 3xy' + 4y = 0$ in the form $y_1(x) = x^\alpha$.
(b) Find the second solution (to the homogeneous equation) in the form $y_2(x) = y_1(x)u(x)$ by deriving the corresponding differential equation for $u(x)$ and solving it.
(c) Find the general solution to the given inhomogeneous equation.

a) $y_1 = x^\alpha \qquad y_1' = \alpha x^{\alpha-1}, \quad y_1'' = \alpha^2 x^{\alpha-2}$

$\rightarrow x^2(\overset{\alpha(\alpha-1)}{\cancel{\alpha^2}} x^{\alpha-2}) - 3x(\alpha x^{\alpha-1}) + 4(x^\alpha) = 0$

$(\alpha^2)x^\alpha - (3\alpha)x^\alpha + 4(x^\alpha) = 0$

$x^\alpha(\alpha^2 - 3\alpha + 4) = 0$

$\alpha = \dfrac{3 \pm \sqrt{-7}}{2}$

2

$*\ y_1 = x^\alpha$
$y_1' = \alpha x^{\alpha-1}$
$y_1'' = \alpha^2 x^{\alpha-2}$ $\alpha(\alpha-1)$

b) $y_2 = y_1 u$

$y_2' = y_1' u + y_1 u'$

$y_2'' = y_1'' u + y_1' u' + y_1' u' + y_1 u''$

$= y_1'' u + 2y_1' u' + y_1 u''$

$\rightarrow x^2(y_1'' u + 2y_1' u' + y_1 u'') - 3x(y_1' u + y_1 u') + 4(y_1 u)$

$= \cancel{x^2 y_1'' u} + 2x^2 y_1' u' + x^2 y_1 u'' - \cancel{3xy_1' u} - 3xy_1 u' + \cancel{4y_1 u}$?

$= u(\underbrace{x^2 y_1'' - 3xy_1' + 4y_1}_{=0}) + u'(2x^2 y_1' - 3xy_1) + u''(x^2 y_1) = 0$

Problem 4. (20 points) Determine the general (real-valued) solution to the system of differential equations $\mathbf{y}' = A\mathbf{y}$, where $A = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}$

$3+2$ 15 $(-2+i)^2 = 4-1 = 3 - 4i$ ~~$-8+4i$~~

$$\lambda^2 + 4\lambda + 5 \to \lambda = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm \frac{2i}{2} = \underline{-2 \pm i}$$

$$(-1-(-2+i))v_1 + 2v_2 = 0$$

$$(1-i)v_1 + 2v_2 = 0$$

$$\to \vec{v} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$(-1)(2) + (-3-(-2+i))(-1-i)$$
$$= -2 + (-1+i)(-1-i)$$
$$= -2 + 1 - i^2 = -2+1+1 = 0 \checkmark$$

$$z = e^{(-2+i)t}\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= e^{-2t}(\cos t + i \sin t)\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= e^{-2t}\left(\cos t\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\left(\cos t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin t\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)\right)$$

$$= e^{-2t}\left[\begin{pmatrix} 2\cos t \\ -\cos t + \sin t \end{pmatrix} + i\begin{pmatrix} 2\sin t \\ -\cos t - \sin t \end{pmatrix}\right]$$

$$\to y = e^{-2t}\left[C_1\begin{pmatrix} 2\cos t \\ -\cos t + \sin t \end{pmatrix} + C_2\begin{pmatrix} 2\sin t \\ -\cos t - \sin t \end{pmatrix}\right]$$