Math 33B-3 Yeliussizov. Midterm 1

Exam time: 12:00-12:50 am, October 21, 2016

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Discussion section (NEITHALATH 3A Tue, 3B Thu; LEE 3C Tue, 3D Thu): 3A

There are 4 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

Problem 1 (20 pt)	Problem 2 (20 pt)	Problem 3 (20 pt)	Problem 4 (20 pt)	Total (80 pt)
7	17	16	18	58

Problem 1. (20 points) Consider the differential equation $y' = y - xy^2$
(a) Sketch its direction field
(b) Given $y(x_0) = 0$ for some x_0, is it then true that $y(x) = 0$ for all real x? (Justify your answer)
(b) Find an appropriate substitution or change of variables so that the equation transforms into a linear or a separable equation
(c) Find explicitly all solutions of the given equation

a)

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$y=0 \to y'=0$
$y=1 \to y' = 1-x$
$y=2 \to y' = 2-4x$
$y=3 \to y' = 3-9x$
$y=-1 \to y' = -1-x$
$y=-2 \to y' = -2-4x$
$y=-3 \to y' = -3-9x$

b) Yes, b/c we see from dir. field that when $y=0$, the $y'=0$, thus y will never change and continue to be 0 for all $x \in \mathbb{R}$. 1

c) $y = xv$

$dy = x\,dv + v\,dx$ $\qquad \dfrac{dy}{dx} = y - xy^2$

$$\frac{x\,dv + v\,dx}{dx} = xv - x(xv)^2 = xv - x^3v^2$$

$$x\frac{dv}{dx} + v = xv - x^3v^2$$

$$x\frac{dv}{dx} = x(v-1) - x^3v^2$$

$$\frac{dv}{dx} = (v-1) - x^2v^2 = v - x^2v^2 - 1$$

1

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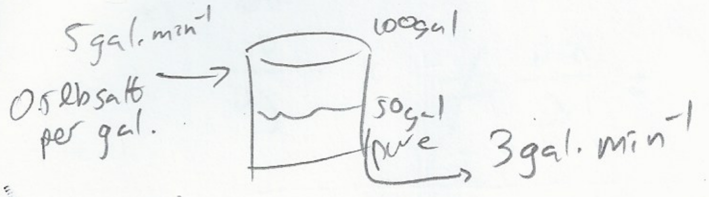
Problem 2. (20 points) A 100-gal tank initially contains 50 gal of pure water. Salt-water solution containing 0.5 lb salt for each gallon of water begins entering the tank at a rate of 5 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the salt-water to leave the tank at a rate of 3 gal/min. What is the salt content (lb) in the tank at the (first) moment when the tank is full?



Tank full at

$$\frac{100-50}{5-3} = 25 \text{ min.}$$

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$S(t)$ = salt in tank, find $S(25)$

$$\frac{dS}{dt} = \text{rate in} - \text{rate out} = \text{concentration} \times \text{liquid} - \text{concentration} \times \text{liquid}$$

$$= \frac{0.5\text{ lb}}{\cancel{\text{gal}}} \cdot \frac{5\ \cancel{\text{gal}}}{\text{min}} - \frac{S(t)\text{ lb}}{V(t)\text{ gal.}} \cdot \frac{3\text{ gal.}}{\text{min}}$$

$$= 2.5 - \frac{3S(t)}{50+2t} \quad ✓ 5 \quad = 2.5 - \left(\frac{3}{50+2t}\right) S(t) = S'(t)$$

$$S_h = e^{-\frac{3}{2}\int \frac{dt}{t+25}} = e^{-\frac{3}{2}\ln(t+25)} = (t+25)^{-3/2} \quad ✓ 5$$

$$S = v S_h \qquad v' = \frac{f(t)}{S_h(t)}$$

$$v' = 2.5(t+25)^{3/2} \rightarrow v = 2.5\,\frac{\sqrt{t+25}}{1/2} = 5\sqrt{t+25} + c$$

$$\rightarrow S = \left(5\sqrt{t+25} + c\right)(t+25)^{-3/2} = \frac{5}{t+25} + \frac{c}{(t+25)^{+3/2}} \quad 2$$

$S(0) = 25$

$$\rightarrow 25 = \frac{5}{25} + \frac{c}{125} = \frac{25+c}{125} \rightarrow c = 600$$

$$S(t) = \frac{5}{t+25} + \frac{600}{(t+25)^{3/2}}$$

$$\boxed{S(25) = \frac{5}{50} + \frac{600}{50^{3/2}}}$$

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Problem 3. (20 points) Consider the differential equation $(xy-1)dx + (x^2 - xy)dy = 0$

(a) Show that it is not exact

(b) Find its integrating factor $\mu = \mu(x)$ if it depends on x only

(c) Find its general solution using the integrating factor μ

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$P = (xy-1)$

$Q = (x^2 - xy)$,

a) $\frac{\partial P}{\partial y} = x \neq \frac{\partial Q}{\partial x} = 2x - y \longrightarrow$ not exact 4

b) $h(x) = \frac{1}{Q}(\partial_y P - \partial_x Q) = \frac{1}{x^2 - xy}(x - 2x + y)$

$= \frac{y-x}{x(x-y)} = \frac{-\cancel{(x-y)}}{x\cancel{(x-y)}} = -\frac{1}{x} = h(x)$ 8

$\mu(x) = e^{\int h(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$

c) $\omega \cdot \mu = (y - \frac{1}{x})dx + (x - y)dy = 0 \rightarrow \frac{\partial P}{\partial y} = 1,\ \frac{\partial Q}{\partial x} = 1 \rightarrow$ exact

supposed to use w Q (x, y)!

$F(x,y) = \int P(x,y)dx + \phi(y) = \int (y - \frac{1}{x})dx + \phi(y) = xy - \ln x + \phi(y)$

$\frac{\partial F}{\partial y} = Q(x,y) = \frac{\partial}{\partial y}(xy - \ln x) + \phi'(y) = x + \phi'(y) = x^2 \cancel{- xy}$ $x - y$

$\cancel{x} + \phi'(y) = \cancel{x} - y$

$\phi'(y) = x^2 - xy - x = (x^2 - x) - xy = \frac{d\phi}{dy}$

$\int d\phi = \int [(x^2 - x) - xy]\,dy$

$\rightarrow \phi = (x^2 - x)y - \frac{xy^2}{2} + C$

$F(x,y) = xy - \ln x + (x^2 - x)y - \frac{xy^2}{2} + C = 0$

need to solve explicitly for y

$\frac{\phi'(y)}{dy} = -y \rightarrow \phi(y) = -\frac{y^2}{2} + c$

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$F(x,y) = xy - \ln x - \frac{y^2}{2} = C$

$y^2 - 2xy + 2\ln|x| = C$

$y = \frac{2x \pm \sqrt{4x^2 - 8\ln|x| + C}}{2} = x \pm \sqrt{x^2 - 2\ln|x| + C}$

Problem 4. (20 points) Consider the autonomous equation $y' = (y-1)^2(y+1)$
(a) Find its equilibrium points
(b) Draw a phase diagram and describe asymptotically stable and unstable points
(c) Sketch the equilibrium solutions in the xy-plane. These solutions divide the plane into regions. Sketch at least one solution trajectory in each of these regions.
(d) Find $\lim_{x\to\infty} y(x)$ for the solution y satisfying the initial condition $y(0) = 0.5$.

a) eq. pts. $\Rightarrow$ $y=1,\ y=-1$

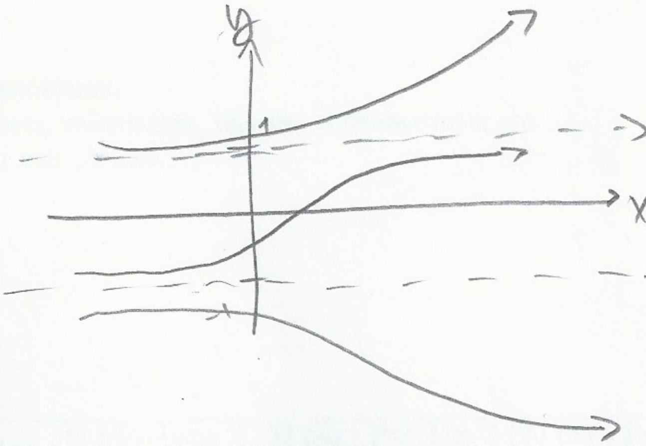
b) $\frac{\partial F}{\partial y}$ $(y^2-2y+1)(y+1)$ 3/5

$= y^3+y^2-2y^2-2y+y+1 = y^3-y^2-y+1$

$\frac{\partial F}{\partial y} = 3y^2-2y-1 \to$ @ $y=1 \to 0$ $\to$ indeterminate

@ $y=-1 \to\ >0$ $\to$ unstable

c)



d) The limit is equal to 1 based on the region the sol'n is in.

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