1. (20 points) Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = x/(1+2y), \quad y(-1) = 0.$$

$\int (1+2y)\,dy = \int x\,dx$

$y^2 + y = \frac{1}{2}x^2 + C$ $\qquad 0^2 + 0 = \frac{1}{2}(-1)^2 + C$

$0 = \frac{1}{2} + C$

$y^2 + y = \frac{1}{2}x^2 - \frac{1}{2}$ $\qquad C = -\frac{1}{2}$

$2y^2 + 2y - (x^2 - 1) = 0$

$y = \frac{-2 \pm \sqrt{4 - 4(2)(1-x^2)}}{4} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 2 + 2x^2}$

$y = -\frac{1}{2} \pm \frac{1}{2}\sqrt{2x^2 - 1}$

The solution to this initial value problem must contain the initial value $y(-1) = 0$, so the solution must be $\boxed{y = -\frac{1}{2} + \frac{1}{2}\sqrt{2x^2-1}}$

~~must contain the initial value $y(-1)=0$, so the interval of existance is~~

$2x^2 - 1$ must be ≥ 0 $\qquad 2x^2 - 1 \geq 0$

$2x^2 \geq 1$

$x^2 \geq \frac{1}{2}$

$x \in (-\infty, -\sqrt{\frac{1}{2}}) \cup (\sqrt{\frac{1}{2}}, \infty)$

Since the interval of existance must contain the initial value $y(-1) = 0$, it has to be $\boxed{x \in (-\infty, -\sqrt{\frac{1}{2}})}$

2. (20 points) Find the solution of the initial value problem.

$$\frac{(1+t^2)y' + 4ty}{1+t^2} = \frac{(1+t^2)^{-2}}{1+t^2}, \quad y(1) = 0.$$

$$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3} \qquad a(t) = -\frac{4t}{1+t^2}$$

$$u = \cancel{y}e^{-\int a(t)\,dt}$$

$$= e^{-\int -\frac{4t}{1+t^2}dt}$$

$$= e^{2\ln|1+t^2|}$$

$$= (1+t^2)^2$$

$$(uy)' = (1+t^2)^{-3} \cdot u$$

$$\int ((1+t^2)^2 y)' = \int (1+t^2)^{-1}$$

$$(1+t^2)^2 y = \arctan t + C$$

$$(1+1^2)^2 \cdot 0 = \arctan 1 + C$$

$$0 = \frac{\pi}{4} + C$$

$$C = -\frac{\pi}{4}$$

$$\frac{\cancel{(1+t^2)^2}y}{\cancel{(1+t^2)^2}} = \frac{\arctan t - \frac{\pi}{4}}{(1+t^2)^2}$$

$$\boxed{y = \frac{\arctan t - \frac{\pi}{4}}{(1+t^2)^2}}$$

✓ 20

3. (20 points) Find the integrating factor to make the following equation into an exact equation. Then find the general solution. (If you remember the integrating factor, you can use it directly.)

$$(x^2y^2 - 1)ydx + (1 + x^2y^2)xdy = 0.$$

20/20

i.f. $= \frac{1}{xy}$

Multiply by integrating factor: $\left(xy^2 - \frac{1}{x}\right)dx + \left(\frac{1}{y} + x^2y\right)dy = 0$ ✓

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2xy$ so this is exact

$\int \frac{\partial F}{\partial x} = \int xy^2 - \frac{1}{x}\,dx$ $\qquad \frac{\partial F}{\partial y} = \frac{1}{y} + x^2y$

$F(x,y) = \frac{1}{2}x^2y^2 - \ln|x| + \phi(y)$ ✓

$\frac{\partial F}{\partial y} = \cancel{x^2y} + \phi'(y) = \frac{1}{y} + \cancel{x^2y}$ $\qquad \int \phi'(y) = \int \frac{1}{y}\,dy$

$\phi(y) = \ln|y|$ ✓

$F(x,y) = \frac{1}{2}x^2y^2 - \ln|x| + \ln|y|$

$$\boxed{\frac{1}{2}x^2y^2 - \ln|x| + \ln|y| = C}$$ ✓

good!

4. (20 points) Suppose that x is a solution to the initial value problem

$x' = x - t^2 + 2t$ $\qquad x' = \cancel{\dfrac{x^3 - x}{1 + t^2 x^2}}, \quad x(0) = 1.$ $\qquad \frac{19}{20}$

Show that $x(t) > t^2$ for all t for which x is defined.

We can see that $x(t) = t^2$ is a solution to this diff. equation:

$x'(t) = 2t \qquad 2t = t^2 - t^2 + 2t$

$2t = 2t$

It has an initial value of $x(0) = 0^2 = 0$ at $t = 0$.

need: $\frac{\partial f}{\partial x} = 1$ $\quad -2$

Since the function itself, $f(t) = x - t^2 + 2t$, and its derivative $\cancel{f'(t) = -2t + 2}$, are both continuous on the whole R^2 plane, and the initial value is contained within that region, the Uniqueness theorem is satisfied: There can only be one solution to initial value problems in this region.

Thus, the solution curves cannot cross. Since the initial value $x(0) = 1$ is greater than the initial value of $x(t) = t^2$ $(x(0) = 0)$, all other solutions to the diffrential equation must be greater than $x(t) = t^2$, so $x(t) > t^2$.

5. (20 points) Find the general solution for the following differential equation.

$$4y'' + 4y' + y = 0.$$

$4\lambda^2 + 4\lambda + 1 = 0$

$(2\lambda + 1)^2 = 0$

$2\lambda + 1 = 0$

$\lambda = -\frac{1}{2}$

Since there is only one solution to the quadratic, the fundamental solutions to the diff. equation is in the form

$\{e^{\lambda t}, te^{\lambda t}\}$

so the fundamental solutions are:

$\{e^{-\frac{1}{2}t}, te^{-\frac{1}{2}t}\}$

and the general form is any linear combination of these fundamental solutions:

$$\boxed{y = C_1 e^{-\frac{1}{2}t} + C_2 t e^{-\frac{1}{2}t}}$$