Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed.

For instructor use only

Page	Points	Score
2	10	8
3	10	10
4	10	10
5	10	10
Total:	40	38

x^{16}

1. (a) [5 pts] Draw equilibrium solutions, and the general shape of solutions between them, for the autonomous differential equation

$\frac{3\pm\sqrt{9-4(-2)}}{2} = \frac{3\pm\sqrt{17}}{2}$ $\quad \frac{3\pm 4}{2}$ $\quad \frac{7}{2}$ $\quad -\frac{1}{2}$ $\quad 3.5$ $\quad -0.5$

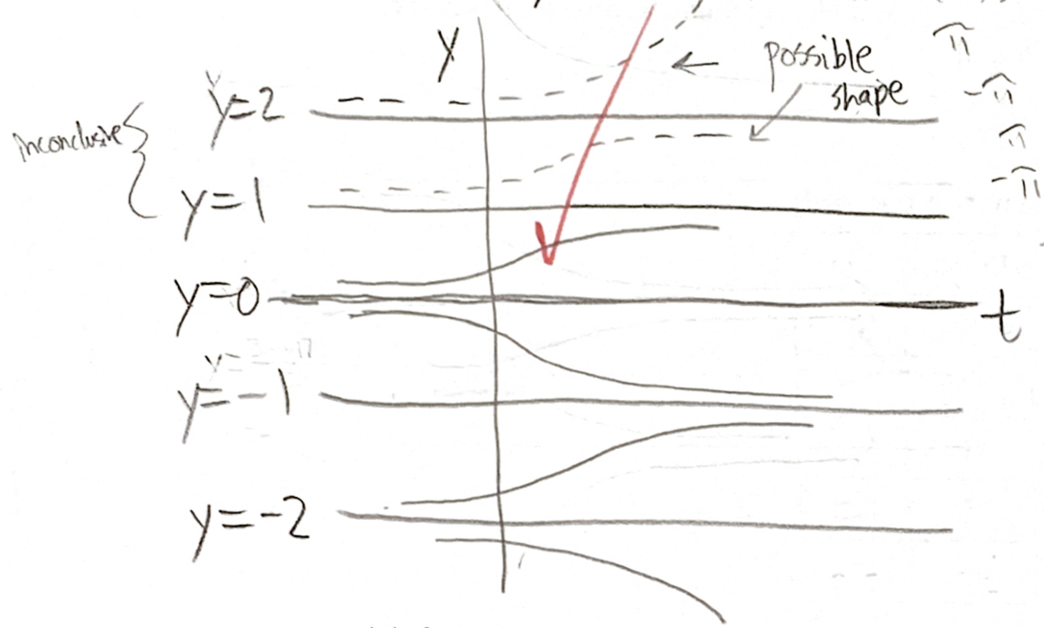
$$y' = (y^2 - 3y + 2)\sin(\pi y).$$

$(y-2)(y-1)$

Include at least 5 equilibrium lines near the origin.

$f(y) = (y^2-3y+2)\sin(\pi y)$ $\qquad y = -2, -1, 0, 1, 2$

$f'(y) = (2y-3)\sin(\pi y) + \pi\cos(\pi y)(y^2-3y+2)$



$-\pi$ $\quad \pi$ $\quad -\pi$ $\quad -\pi$ $\quad 4+6+2$ $\quad 1+3+2$ $\quad 2$ $\quad 0$ $\quad 2$

y	$f'(y)$
-2	12π
-1	-6π
0	2π
1	0
2	0
3	-2π
4	6π

(b) [5 pts] Describe the stability of all equilibrium points for the differential equation above. There should be some special cases, and then some statements that hold for entire families of equilibrium points.

SO DON'T USE f'

-2

The roots $y=1, 2$ are both special cases as $f'(y)$ for them is equal to 0, so we cannot definitively state whether they are stable or not.

Besides those roots, even integers are unstable, while odd integers are asymptotically stable.

2. (a) [8 pts] Find the equations of motion (ie solve the initial value problem) for a mass of $2kg$ on a spring having spring constant $72kg/s^2$ in a liquid with damping constant $24kg/s$, assuming that the mass has a starting position of $1m$ and velocity of $2m/s$ at time $t = 0$.

$m=2$ kg $\quad k=72$ kg/s^2 $\quad \mu=24$ kg/s $\quad x(0)=1$

$\dot{x}(0)=2$

$$2x'' + 24x' + 72x = 0$$

$$x'' + 12x' + 36x = 0$$

Characteristic poly. $\Rightarrow \lambda^2 + 12\lambda + 36 = 0$

$$(\lambda+6)(\lambda+6)$$

$$\lambda = -6$$

$$x(t) = C_1 e^{-6t} + C_2 t e^{-6t} \qquad x'(t) = -6C_1 e^{-6t} + C_2 e^{-6t} - 6C_2 t e^{-6t}$$

$$x(0) = 1 = C_1 \qquad x'(0) = 2 = -6C_1 + C_2$$

$$C_1 = 1 \qquad C_2 = 6C_1 + 2 = 8$$

$$\Rightarrow \boxed{x(t) = e^{-6t} + 8te^{-6t}}$$

$$A = \sqrt{a^2+b^2} = \sqrt{1+64} = \sqrt{65}$$

$w = -6$ rad/sec

phase $= 0$

(b) [2 pts] How fast is the spring moving at time $t = 1$?

$$x(t) = e^{-6t} + 8te^{-6t}$$

$$x'(t) = -6e^{-6t} + 8e^{-6t} - 48te^{-6t}$$

$$x'(1) = -6e^{-6} + 8e^{-6} - 48e^{-6} = -46e^{-6}$$

$\boxed{46e^{-6} \text{ m/s in direction opposite to starting direction}}$

3. [10 pts] Find the general solution to the differential equation

$$y'' + y = \overset{g}{\sec^2 t}$$

(You will need the formula $\int \sec t dt = \ln|\sec t + \tan t|$)

$g(t)=\sec^2 t$

$y''+y=0$

$\lambda^2+1=0$

$\lambda = \pm i$

$\frac{\pm\sqrt{0-4(1)(1)}}{2} = \frac{\sqrt{-4}}{2} = \frac{2i}{2} = i$

$z(t) = e^{it} = \cos t + i\sin t$

$y_h = C_1\cos t + C_2 \sin t$

$y_1 = \cos t \quad y_2 = \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$v_1' = \frac{-y_2 g(t)}{W} = \frac{-\sin t(\sec^2 t)}{1} = -\sin t(\sec^2 t)$$

$$v_2' = \frac{y_1 g(t)}{W} = \cos t \sec^2 t$$

$u = \cos t \quad du = \sin t \quad u^{-2}$

$$v_1 = -\int \sin t \sec^2 t\, dt = -\int \frac{\sin t}{\cos^2 t} dt = \int \frac{1}{u^2} du = -u^{-1} = -(\cos t)^{-1} = -\sec t$$

$$v_2 = \int \cos t \sec^2 t\, dt = \int \frac{1}{\cos t} dt = \int \sec t\, dt = \ln|\sec t + \tan t|$$

$$y_p = v_1 y_1 + v_2 y_2 = (-\sec t)(\cos t) + (\ln|\sec t + \tan t|)(\sin t)$$

$$= -1 + \sin t \ln|\sec t + \tan t|$$

$$y = y_h + y_p = \boxed{C_1\cos t + C_2 \sin t + \sin t \ln|\sec t + \tan t| - 1}$$

4. [10 pts] Suppose that y_1 and y_2 are two solutions to the second order homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Prove that the Wronskian of y_1 and y_2, denoted $W(t)$, satisfies its own differential equation

$$W'(t) = -p(t)W(t).$$

(You cannot use the formula $W(t) = W(t_0)e^{\int p(t)dt}$ in this problem - in fact this problem is a part of the proof of that formula) *Hint: Differentiate the standard formula for W, and use the fact that both y_1 and y_2 satisfy the homogeneous equation to make a substitution for the second derivatives in your expression*

$$W = y_1 y_2' - y_1' y_2$$

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2 = y_1 y_2'' - y_1'' y_2$$

We know that: since both y_1 and y_2 satisfy the homog. eq.

$$y_2'' = -p(t)y_2' - q(t)y_2$$

$$y_1'' = -p(t)y_1' - q(t)y_1$$

Now substitute

$$W' = y_1(-p(t)y_2' - q(t)y_2) - y_2(-p(t)y_1' - q(t)y_1)$$

$$W' = -p(t)\, y_1 y_2' + p(t)\, y_1' y_2$$

$$= p(t)\underbrace{(y_1' y_2 - y_1 y_2')}_{-W}$$

$$\Rightarrow \quad W' = -p(t)\, W \quad \checkmark$$