1. [4 pts] Find the general solution to the differential equation

$$y' = \frac{x + \cos x}{1 + 2y}.$$

You may leave your answer in implicit form.

$$\frac{dy}{dx} = \frac{x + \cos x}{1+2y}$$

$$(1+2y)\,dy = (x+\cos x)\,dx$$

$$\int 1\,dy + \int 2y\,dy = \int x\,dx + \int \cos x\,dx$$

$$y + y^2 = \frac{1}{2}x^2 + \sin x + C$$

The solution is :

$$F(x,y) = y^2 + y - \frac{1}{2}x^2 - \sin x = C, \quad C \text{ is arbi. constant}$$

0.5 lb/gal.

1.25
4T
$\frac{5}{4}$
$\frac{}{10}$
$\frac{5}{20}$
$\frac{25}{20}$

2. [10 pts] A tank initially contains 100gal of salt-water containing .5lb of salt per gallon of water. At time zero, pure water is poured-in at 3gal/min, while simultaneously a drain is opened at the bottom of the tank allowing the mixed salt-water to leave at 2gal/min. What will be the amount of salt in the tank when it has 200gal of water?

Let $x(t)$ be the amount of salt in the tank at any time t.

$x(0) = 0.5 \times 100 = 50\text{ lb}$

$\frac{dx}{dt} = \text{rate in} - \text{rate out}$

rate in $= 0 \times 3$

rate out $= \frac{x}{V} \times 2$

Let $V(t)$ be the volume of salt-water at any time t.

$V(t) = 100 + (3-2)t$

$= 100 + t$

$\frac{dx}{dt} = 0 - \frac{x}{100+t} \times 2$

$\frac{dx}{dt} = -\frac{2x}{100+t}$

$x' + \frac{2}{100+t} \cdot x = 0$

$u = 100+t$

$du = dt$

Let $u(t) = e^{-\int -\frac{2}{100+t}\,dt}$

$\int \frac{2}{100+t}\,dt$

$= e^{2\ln|100+t|}$

$= \int \frac{2}{u}\,du$

$= (100+t)^2$

$= 2\int \frac{du}{u}$

$= 2\ln|u|$

$= 2\ln|100+t|$

$(100+t)^2 x' + (100+t) \cdot 2x = 0$

$\left[(100+t)^2 \cdot x\right]' = 0$

$(100+t)^2 \cdot x = C$

$x = \frac{C}{(100+t)^2}$, C is arbi. constant

Plug in $x(0) = 50$

$x(0) = \frac{C}{(100)^2} = 50$

$C = 5 \times 10^5$

$x(t) = \frac{5\times10^5}{(100+t)^2}$

$(200)^2$

2×10^2

4×10^4

When $V(t) = 200$, $t = 100$

$x(100) = \frac{5\times10^5}{4\times10^4}$

$= \frac{5}{4} \times 10$

$= 12.5\text{ lb}$

3. In this problem we explore the notion of using an integrating factor $\mu(y)$ that depends on y only in order to make the differential form $P(x,y)dx + Q(x,y)dy$ exact.

   (a) [6 pts] Suppose that there exists an integrating factor $\mu(y)$ making the new differential form $\mu(y)P(x,y)dx + \mu(y)Q(x,y)dy$ exact. Prove that we must have

$$\frac{d\mu}{dy} = g\mu$$

   where g is the function

$$g(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

To make the new differential form exact,

we must have $\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$ Why? b/c $\frac{\partial \mu}{\partial x} = 0$

$$\frac{\partial}{\partial y}\mu \cdot P + \frac{\partial}{\partial y}P \cdot \mu = \mu \cdot \frac{\partial}{\partial x} Q$$

5

$$P \cdot \frac{\partial \mu}{\partial y} = \mu \cdot \frac{\partial Q}{\partial x} - \mu \cdot \frac{\partial P}{\partial y}$$

$$\frac{\partial \mu}{\partial y} = \underbrace{\frac{1}{P} \cdot \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{g(y)} \cdot \mu$$

$$\frac{d\mu}{dy} = g(y)\mu, \quad g(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

   (b) [4 pts] Show that defining

$$\mu(y) := e^{\int g(y)dy}$$

   satisfies the differential equation $\frac{d\mu}{dy} = g\mu$, and so this should be our definition for $\mu(y)$ in such a case.

$$LHS = \frac{d\mu}{dy} = e^{\int g(y)\,dy} \cdot g(y)$$

$$RHS = g(y) \cdot \mu = e^{\int g(y)\,dy} \cdot g(y)$$

$$\Rightarrow LHS = RHS$$

$$\mu(y) = e^{\int g(y)\,dy}$$

$-2x - 2y$ ...

(c) [10 pts] Solve the differential equation $(y^2 + 2xy)dx - x^2 dy = 0$, assuming that there is an integrating factor $\mu(y)$ that depends on y only (you may use previous parts of the problem, even if you were not able to prove them).

$P = y^2 + 2xy$

$Q = -x^2$

$g(y) = \frac{1}{y^2+2xy} \cdot \left(-2x - (2y+2x)\right)$

$= \frac{1}{y^2+2xy} \cdot (-4x - 2y)$

$= \frac{-2 \cdot \cancel{(2x+y)}}{y\cancel{(y+2x)}}$

$= -\frac{2}{y}$

$\mu(y) = e^{\int -\frac{2}{y} dy}$

$= e^{-2\int \frac{1}{y} dy}$

$= e^{-2\ln|y|}$

$= |y|^{-2} = \frac{1}{y^2}$

$\mu P = 1 + \frac{2x}{y}$

$\mu Q = -\frac{x^2}{y^2}$

$\frac{\partial}{\partial y}(\mu P) = 2x \cdot \left(-\frac{1}{y^2}\right)$

$\frac{\partial}{\partial x}(\mu Q) = \left(-\frac{1}{y^2}\right) \cdot 2x$

$\Rightarrow$ exact !

$\mu P = \frac{\partial F}{\partial x}$

$F = \int 1 + \frac{2x}{y} dx$

$= x + \frac{x^2}{y} + \phi(y)$

$\frac{\partial F}{\partial y} = 0 + x^2 \cdot \left(-\frac{1}{y^2}\right) + \phi'(y) = -\frac{x^2}{y^2}$

$\phi'(y) = 0 \Rightarrow \phi(y) = 0$

Thus, $F(x,y) = x + \frac{x^2}{y} = C$

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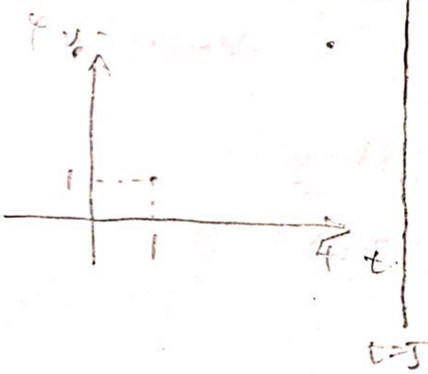
4. Suppose that y is a solution to the initial value problem

$$(t-5)y' = (y-3)\cos(\cos(ty)), \quad y(1) = 1$$

and that y is actually defined for all t.

(a) [3 pts] Is it possible that $y(4) = 4$? Explain (quote a theorem from class, or explain why it does not apply).



$$y' = \frac{(y-3)\cos(\cos(ty))}{t-5} = f(t,y)$$

$f(t,y)$ is not continuous when $t=5$ but continuous everywhere else.

By existence theorem,

We can find a rectangle R containing $(1,1)$ in which f is continuous.

And point $(4,4)$ is on the left side of $t=5$, so the rectangle can contain $(4,4)$

The answer is yes.

−1

(b) [3 pts] Is it possible that $y(6) = 6$? Explain (quote a theorem from class, or explain why it does not apply).

No. By (a), $(6,6)$ is on the right side of $t=5$. ~~By uniqueness theorem, solution~~

~~The rectang~~

−2

We cannot find a rectangle R containing both $(1,1)$ and $(6,6)$ in which f is continuous.