(1) (12 points) Consider the equation

$$x''' + x'' + 4x' + 4x = 0.$$

(a) Find the general solution to this equation.

(b) Find the solution to the equation satisfying the initial conditions

9

$$x(0) = 1, \quad x'(0) = -1, \quad \text{and} \quad x''(0) = \alpha,$$

where α denotes a real constant.

(c) Find the value of the constant $\alpha \in (-\infty, \infty)$ for which the solution you found in part (b) approaches zero as $t \to \infty$.

a) $\lambda^3 + \lambda^2 + 4\lambda + 4 = 0$

$\lambda^2(\lambda+1) + 4(\lambda+1) = 0$

$(\lambda^2+4)(\lambda+1) = 0$

$\lambda = \pm 2i, -1$

$\boxed{x(t) = c_1 e^{-t} + c_2\cos(2t) + c_3\sin(2t)}$

b) $x(0) = 1 = c_1 + c_2$

$x' = -c_1 e^{-t} - 2c_2\sin(2t) + 2c_3\cos(2t)$

$x'(0) = -1 = -c_1 - 2c_3$

$x'' = c_1 e^{-t} - 4c_2\cos(2t) - 4c_3\sin(2t)$

$x''(0) = \alpha = c_1 - 4c_2$

$c_1 = 4c_2 + \alpha$

$1 = 4c_2 + \alpha + c_2$

$1 - \alpha = 5c_2$

$c_2 = \frac{1-\alpha}{5}$

$1 = c_1 + \frac{1-\alpha}{5}$

$c_1 = \frac{5}{5} - \frac{1-\alpha}{5} = \frac{4-\alpha}{5}$

$-1 = -c_1 - 2c_3$

$-\frac{5}{5} = -\frac{4-\alpha}{5} - 2c_3$

$-\frac{5}{5} + \frac{4-\alpha}{5} = -2c_3$

$c_3 = \frac{-1-\alpha}{10}$

-1

$x(t) = \underbrace{\left(\frac{1-\alpha}{5}\right)e^{-t}}_{x_1} + \underbrace{\left(\frac{4-\alpha}{5}\right)\cos(2t)}_{x_2} + \underbrace{\left(\frac{-1-\alpha}{10}\right)\sin(2t)}_{x_3}$

c) x_1 naturally goes to 0 as $t \to \infty$

-2 $\boxed{\alpha = 4}$ for $t \to \infty$

$\frac{8-2\alpha}{10} = -\frac{1-\alpha}{10}$

$8 - 2\alpha = -1 - \alpha$

$9 = \alpha$

9

(2) (10 points) Find the general solution to the differential equation

$$x'' - 5x' + 4x = \sin(t) + 8.$$

~~method~~ of annihilators!

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 4)(\lambda - 1) = 0$

$\lambda = 4, 1$

$x_c = C_1 e^{4t} + C_2 e^{t}$

Method of Annihilators!

$\left(\frac{d^2}{dt^2} + 1\right)(\sin t) = 0$

$\frac{d}{dt}(8) = 0$

$x'' - 5x' + 4x = \sin t + 8$

$\left(\frac{d^2}{dt^2} - 5\frac{d}{dt} + 4\right)x = \sin t + 8$

$\left(\frac{d}{dt}\right)\left(\frac{d^2}{dt^2} + 1\right)\left(\frac{d}{dt} - 4\right)\left(\frac{d}{dt} - 1\right) = 0$

$\lambda \cdot (\lambda^2 + 1)(\lambda - 4)(\lambda - 1) = 0$

$\lambda = 0, \pm i, 4, 1$

$x(t) = C_1 e^{4t} + C_2 e^{t} + \underbrace{C_3 + C_4 \cos t + C_5 \sin t}_{x_p}$ ✓

$x_p = C_3 + C_4 \cos t + C_5 \sin t$

$x_p' = -C_4 \sin t + C_5 \cos t$

$x_p'' = -C_4 \cos t - C_5 \sin t$

$$\boxed{x_{gen} = C_1 e^{4t} + C_2 e^{t} + 2 + \cancel{\tfrac{1}{8}} \sin t}$$

-1

$\sin t + 8 = -C_4 \cos t - C_5 \sin t + 5C_4 \sin t - 5C_5 \cos t + 4C_3 + 4C_4 \cos t + 4C_5 \sin t$

$\sin t + 8 = 3C_5 \sin t + 3C_4 \cos t + 4C_3 + 5C_4 \sin t - 5C_5 \cos t$

$4C_3 = 8$

$C_3 = 2$

$1 = 3C_5 + 5C_5$

$1 = 8C_5$

$C_5 = \frac{1}{8}$

$0 = 3C_4 - 5C_4$

$C_4 = 0$

$-\frac{\sin t}{t}$

$-\frac{(\sin^2 + \cos^2)}{t^2} = -\frac{1}{}$

(3) (14 points)

Consider the equation

$$tx'' + 2x' + tx = 1 \quad \text{with} \quad t > 0.$$

(a) Verify that $\phi_1(t) = \frac{\sin(t)}{t}$ and $\phi_2(t) = \frac{\cos(t)}{t}$ form a fundamental set of solutions to the associated homogeneous equation for $t \in (0, \infty)$.

(b) Find a particular solution to the given inhomogeneous equation.

(c) Write down the general solution to the inhomogeneous equation.

a) $W(\Phi_1, \Phi_2)(t) = \begin{vmatrix} \Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2' \end{vmatrix} = \begin{vmatrix} \frac{\sin t}{t} & \frac{\cos t}{t} \\ \left(\frac{t\cos t - 1}{t^2}\right) & \left(\frac{-t\sin t - 1}{t^2}\right) \end{vmatrix} = \frac{-t\sin^2 t - \sin t - t\cos^2 t + \cos t}{t^3} = \frac{-t - \sin t + \cos t}{t^3} \neq 0$ for all $t > 0$

+3

$\frac{\sin t}{t}: \; t\left(\frac{-\sin t}{t} - \frac{1}{t^2} + \frac{2}{t^2}\right) + 2\left(\frac{\cos t}{t} - \frac{1}{t^2}\right) + \sin t = 0$

$-\sin t - \frac{1}{t} + \frac{2}{2t} + \frac{2\cos t}{t}$

$t\left(\frac{t^2[\cos t - t\sin t] - 2t}{t^2}\right) + 2\left(\frac{t\cos t - 1}{t^2}\right) + t\left(\frac{\sin t}{t}\right) = 0$

$= t(\cos t) - t^2 \sin t - 2 + 2t\cos t - 2 + \sin t$

b) $x_p = v_1 \Phi_1 + v_2 \Phi_2$ $\qquad x'' + \frac{2}{t}x' + x = \frac{1}{t}$ $\qquad f(t) = \frac{1}{t}$ $\qquad$ +5

$v_1 = -\int \frac{\Phi_2 f(t)}{W(\Phi_1, \Phi_2)} = -\int \frac{t^3 \cdot \frac{\cos t}{t} \cdot 1/t}{-t - \sin t + \cos t} dt = \int -\frac{t\cos t}{t + \sin t + \cos t} dt = y(t)$

$v_2 = \int \frac{\Phi_1 f(t)}{W(\Phi_1, \Phi_2)} = \int \frac{\frac{\sin t}{t} \cdot \frac{1}{t} \cdot t^3}{-t - \sin t + \cos t} dt = \int \frac{t\sin t}{-t - \sin t + \cos t} dt = Z(t)$

$\boxed{x_p = y\left(\frac{\sin t}{t}\right) + Z\left(\frac{\cos t}{t}\right)}$

+2

c) $x_{gen} = x_p + x_c = \boxed{C_1 \frac{\sin t}{t} + C_2 \frac{\cos t}{t} + \frac{y\sin t}{t} + \frac{Z\cos t}{t}}$

(4) (14 points)

Consider the differential equation

$$t^2x'' - 2x = t^2 \quad \text{with} \quad t > 0. \tag{1}$$

(a) Verify that $\phi_1(t) = t^2$ is a solution to the associated homogeneous equation

$2t$

$$t^2x'' - 2x = 0. \tag{2}$$

(b) Look for a solution to the inhomogeneous equation (1) of the form $x(t) = v(t)\phi_1(t)$. Plug this into equation (1) and derive a differential equation for v. Solve it.

(c) Write down the general solution to equation (1).

(d) Write down a fundamental set of solutions for equation (2) for $t > 0$.

$\phi_1 = t^2 \quad \phi_1' = 2t \quad \phi_1'' = 2$

a) $t^2(2) - 2(t^2) = 2t^2 - 2t^2 = 0 = 0$ ✓

b) $\phi_2 = v \cdot \phi_1 = vt^2 \qquad \phi_2'' = v''t^2 + 4v't + 2v$

$\phi_2' = v't^2 + 2vt$

$x'' - \frac{2}{t^2}x = 1 \qquad f(t) = 1$

$v''t^2 + 4v't + 2v - \frac{2}{t^2} \cdot vt^2 = 1$

$v''t^2 + 4v't + 2v - 2v = 1$

$t^2v'' = -4v't + 1$

$v'' = -\frac{4}{t}v + \frac{1}{t^2} \qquad u(t) = e^{-\int -\frac{4}{t}dt} = e^{4\ln|t|} = t^4$

$t^4v'' + \frac{4}{1}t^3v = t^2$

$(t^4v') = \int t^2 dt$

$t^4 \cdot v' = \frac{t^3}{3} + A$

$v' = \frac{1}{3t} + \frac{A}{t^4}$

$v = \int \frac{1}{3t} + \frac{A}{t^4} dt$

$\boxed{v = 3\ln|t| - \frac{A}{3t^3} + C}$

c) $\boxed{x = C_1t^2 + t^2[3\ln|t| - \frac{A}{3t^3} + C]}$

d) $\phi_1 = t^2$

$\phi_2 = 3\ln|t| - \frac{A}{3t^3} + C$? $\leftarrow x_p$

-3