33B Visan Midterm 1 October 17th

First Name: [redacted] ID# [redacted]

Last Name: [redacted]

Section: 1E

$$= \begin{cases} 1A & \text{Tuesday with J. Yu} \\ 1B & \text{Thursday with J. Yu} \\ 1C & \text{Tuesday with D. Lichko} \\ 1D & \text{Thursday with D. Lichko} \\ 1E & \text{Tuesday with A. Lin} \\ 1F & \text{Thursday with A. Lin} \end{cases}$$

Rules:

- There are **FOUR** problems for a total of 50 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	Σ
11	8	8	17	44

11

(1) (12 points)

A tank of capacity 1000 gallons initially contains 200 gallons of water in which 40 lbs of sugar are dissolved. Water containing 1 lb of sugar per gallon is pumped into the tank at the rate of 4 gallons per minute. The well-mixed solution is pumped out of the tank at the rate of 2 gallons per minute.

(a) When will the tank be full?

(b) Find the quantity of sugar (measured in lbs) in the tank at any time.

(c) What is the concentration of sugar in the tank at the time the tank is full?

$x(t)$ = sugar in [lbs]

a) V = volume of solution in tank [gallons]

$$\frac{dV}{dt} = V_{in} - V_{out} = 4-2 = 2 \text{ gallons/min}$$

$$V_{tank} = \frac{dV}{dt}\cdot t + V_{initial}$$

$$1000 = 2\cdot t + 200$$

$$800 = 2t$$

$$\boxed{t = 400 \text{ minutes}}$$

b) ① $$\frac{dx}{dt} = 1\cdot 4 - 2\cdot\frac{x}{2t+200} = 4 - \frac{x}{t+100}$$

$$p(t) = \frac{-1}{t+100}$$

$$g(t) = 4$$

find the integrating factor

$$u(t) = e^{-\int \frac{-1}{t+100}dt} = e^{\int \frac{1}{t+100}dt} = e^{\ln|t+100|} = t+100$$

b) ② $$x(t) = \frac{1}{u(t)}\int g(t)u(t)\,dt = \frac{1}{t+100}\int (t+100)4\,dt = \frac{4}{t+100}\left(\frac{t^2}{2} + 100t + c\right) = \frac{2t^2+400t+c}{t+100}$$

$$x(0) = 40$$

$$\frac{\cancel{2\cdot 0} + \cancel{400\cdot 0} + c}{0+100} = 40$$

$$c = 40\cdot 100 = 4000$$

$$\boxed{x(t) = \frac{2t^2+400t+4000}{t+100}}$$

c) tank is full at $t = 400$,

$$x(400) = \frac{2\cdot 400^2 + 400^2 + 4000}{400+100} = \frac{3\cdot 160000 + 4000}{500} = \frac{8800}{500} = \frac{88}{5} \text{ lbs}$$

$$\text{concentration} = \frac{88/5}{1000} = \frac{88}{5000} = \frac{44}{2500} = \boxed{\frac{11}{625}\,\%}$$

−1

(2) (12 points)

(a) Show that the differential equation

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$$(4y + 2x - 2)dx + (6y + 4x - 1)dy = 0$$

is exact on the whole plane $(-\infty, \infty) \times (-\infty, \infty)$.

(b) Find the *explicit* solution to this equation subject to the initial condition $y(0) = 1$.

(c) What is the interval of existence of the solution you found in part (b)?

a) $P(x,y) = 4y + 2x - 2$

is continuous on whole plane + differentiab. +1

$\frac{\partial P}{\partial y} = 4$

is continuous on whole plane

$Q(x,y) = 6y + 4x - 1$

is continuous on whole plane

$\frac{\partial Q}{\partial x} = 4$

is continuous on whole plane

$4 = 4$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on $R = (-\infty,\infty)\times(-\infty,\infty)$

$\therefore$ $\boxed{(4y+2x-2)dx + (6y+4x-1)dy = 0 \text{ is exact on whole plane } R = (-\infty,\infty)\times(-\infty,\infty)}$

b) $(4y+2x-2)dx + (6y+4x-1)dy = 0$

① is an exact equation

$F = \int P(x,y)\,dx$ +6

$= \int 4y + 2x - 2\,dx$

$= 4xy + x^2 - 2x + \phi(y)$

$\frac{\partial F}{\partial y} = Q(x,y)$

b) ② $\frac{\partial F}{\partial y} = 4x + \phi'(y)$

$Q(x) = 6y + 4x - 1$

$\cancel{4x} + \phi'(y) = 6y + \cancel{4x} - 1$

$\phi'(y) = 6y - 1$

$\phi(y) = \int 6y - 1\,dy$

$= 3y^2 - y + C$

$F(x,y) = 4xy + x^2 - 2x + 3y^2 - y = C$

$y(0) = 1$

$\therefore 4\cdot 0\cdot 1 + 0^2 - 2\cdot 0 + 3\cdot 1^2 - 1 = C$

explicit? $C = 3 - 1 = 2$

$\boxed{F(x,y) = 4xy + x^2 - 2x + 3y^2 - y = 2}$

c) $3y^2 + (4x-1)y + (x^2 - 2x - 2) = 0$

$y = \frac{-(4x-1) \pm \sqrt{(4x-1)^2 - 12(x^2-2x-2)}}{6}$

$(4x-1)^2 > 12(x^2 - 2x - 2)$

$16x^2 - 8x + 1 > 12x^2 - 24x - 24$

$4x^2 + 16x + 25 > 0$

y & x are both polynomials

$\therefore \boxed{I = (-\infty, \infty)}$

+1

(3) (8 points)

Solve the initial-value problem

$$\frac{dx}{dt} = 2x - te^{2t} \quad \text{with} \quad x(0) = -2.$$

What is the interval of existence of the solution?

$\frac{dx}{dt} = 2x - te^{2t}, \; x(0) = -2$

$f(t) = 2 \quad g(t) = -te^{2t}$

Find integrating factor

$u(t) = e^{-\int 2\,dt}$

$= e^{-2t}$

$x(t) = \frac{1}{u(t)} \int g(t)\, u(t)\, dt$

$= \frac{1}{e^{-2t}} \int -t\cancel{e^{2t}}\cancel{e^{-2t}}\, dt$

$= e^{2t} \int -t\, dt$

$= \frac{-e^{2t} t^2}{2} + e^{2t} c$

$x(0) = -2$

$\frac{-e^0 \cdot 0}{2} + e^0 \cdot c = -2$

$c = -2$

$x(t) = \frac{-e^{2t} t^2}{2} - 2e^{2t}$

$\boxed{x(t) = -\left(\frac{e^{2t} t^2}{2} + 2e^{2t}\right)}$

Interval of existence

$x(t) = -\left(\frac{e^{2t} t^2}{2} + 2e^{2t}\right)$

only polynomials & exponentials

$\boxed{I = (-\infty, \infty)}$

✓ 8

(4) (18 points)

Consider the differential equation

$$\frac{dx}{dt} = x^3(x-1)e^x.$$

(a) Explain why for any $x_0 \in (-\infty, \infty)$ there exists a unique solution to the equation satisfying the initial condition $x(0) = x_0$, at least on some time interval containing 0.

(b) Sketch the function $f(x) = x^3(x-1)e^x$ and identify the equilibrium points.

(c) Draw a phase diagram and identify the stable and unstable points.

(d) Sketch the equilibrium solutions in the tx plane. These equilibrium solutions divide the tx plane into regions. Sketch at least one solution curve in each of these regions.

(e) For the particular solution with initial condition $x(0) = -3$, what is the limit $\lim_{t\to\infty} x(t)$?

a) $f(x) = x^3(x-1)e^x$

is continuous on $x \in (-\infty, \infty)$, because it contains only polynomials & exponentials

$\therefore$ satisfies Peano's existence theorem

$f'(x) = e^x(x^4 - x) + (4x^3 - 1)e^x$

is continuous on $x \in (-\infty, \infty)$ because it contains only polynomials & exponentials

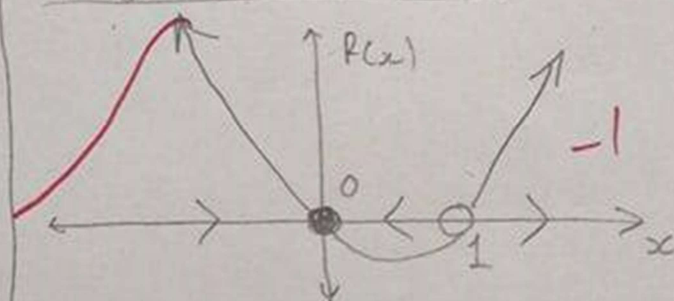
$\therefore$ satisfies Picard's uniqueness theorem on $x \in (-\infty, \infty)$

$\therefore$ For any $x_0 \in (-\infty, \infty)$, there exists a unique solution satisfying $x(0) = x_0$, at least on some time interval containing 0

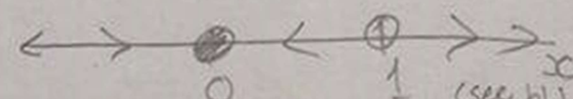
b) $f(x) = x^3(x-1)e^x$

$0 = f(x)$

$\underbrace{x = 0, 1}$ equilibrium points



c)

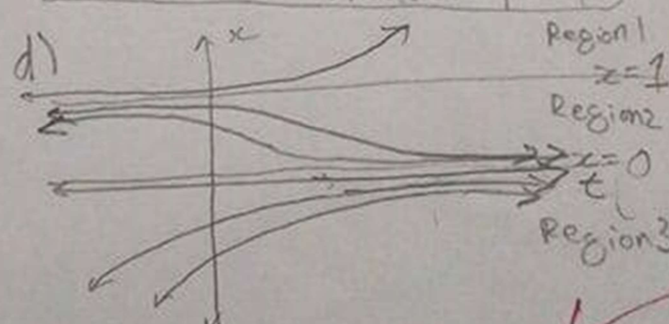


(see b))

x = 0 is a stable point

x = 1 is an unstable point

d)



e) $x(0) = -3$ is in Region 3 $\therefore$

$\lim_{t\to\infty} x(t) = 0$