33B Visan | Midterm 1 | October 17th

First Name: ________

ID#________

Last Name: ________

Section: 1D

- 1A Tuesday with J. Yu
- 1B Thursday with J. Yu
- 1C Tuesday with D. Lichko
- 1D Thursday with D. Lichko
- 1E Tuesday with A. Lin
- 1F Thursday with A. Lin

Rules:

- There are **FOUR** problems for a total of 50 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	Σ
12	11	8	17	48

(1) (12 points)

A tank of capacity 1000 gallons initially contains 200 gallons of water in which 40 lbs of sugar are dissolved. Water containing 1 lb of sugar per gallon is pumped into the tank at the rate of 4 gallons per minute. The well-mixed solution is pumped out of the tank at the rate of 2 gallons per minute.

(a) When will the tank be full?

(b) Find the quantity of sugar (measured in lbs) in the tank at any time.

(c) What is the concentration of sugar in the tank at the time the tank is full?

Let $v(t)$ = gallons of water in tank at t (minutes)

Let $A(t)$ = pounds of sugar in tank at t (minutes)

12

a) $v(t) = 200 + 2t$

rate in = 4

rate out = 2 $4-2=2$

$1000 = 200 + 2t$

$2t = 800$

$t = 400$ min

$\boxed{\text{After 400 minutes}}$

b) rate in $= 4\frac{gal}{min} \cdot \frac{1\ lb}{gal} = 4\frac{lb}{min}$

rate out $= 2\frac{gal}{\ } \cdot \frac{A(t)\ lb}{\ } = \frac{2A(t)}{\ }\frac{lb}{min} = \frac{A(t)}{100+t}\frac{lb}{min}$

(2) (12 points)

(a) Show that the differential equation

$$(4y + 2x - 2)dx + (6y + 4x - 1)dy = 0$$

is exact on the whole plane $(-\infty, \infty) \times (-\infty, \infty)$.

(b) Find the *explicit* solution to this equation subject to the initial condition $y(0) = 1$.

(c) What is the interval of existence of the solution you found in part (b)?

a) $P = 4y + 2x - 2$ P is continuous on whole plane and differentiable wrt y.

$Q = 6y + 4x - 1$ Q is continuous on whole plane and differentiable wrt x.

we need $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

-2

$\frac{\partial P}{\partial y} = 4$, $\frac{\partial Q}{\partial x} = 4$ ⇒ The equation is exact on the whole plane.

b) $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$

$\int P\,dx = F$: $F = 4yx + x^2 - 2x + \phi(y)$

$\frac{\partial F}{\partial y} = Q$: $\frac{\partial F}{\partial y} = 4x + \phi'(y) = 6y + 4x - 1$

$\Rightarrow \phi'(y) = 6y - 1$

$\Rightarrow \phi(y) = 3y^2 - y$

* plug in $y(0) = 1$ to implicit solution

$0 + 0 - 0 + 3(1)^2 - 1 = C \Rightarrow C = 2$

$4yx + x^2 - 2x + 3y^2 - y - 2 = 0$

(3) (8 points)

Solve the initial-value problem

$$\frac{dx}{dt} = 2x - te^{2t} \quad \text{with} \quad x(0) = -2.$$

What is the interval of existence of the solution?

$a = 2\ ,\ f = -te^{2t}$

Int. factor $u = e^{-\int 2\,dt} = e^{-2t}$

$x = \frac{1}{u}\int uf = e^{2t}\int (\cancel{e^{-2t}})(-t\cancel{e^{2t}})\,dt = e^{2t}\int -t\,dt$

$= e^{2t}\left(-\frac{t^2}{2} + c\right)$

$x(0) = -2 \Rightarrow \cancel{e^0}\left(-\frac{0}{2} + c\right) = -2$

$\Rightarrow c = -2$

$$\boxed{x(t) = e^{2t}\left(-\frac{t^2}{2} - 2\right)}$$

$\sqrt{8}$

$\boxed{\text{Interval of existence } I = (-\infty, \infty)}$

(4) (18 points)

Consider the differential equation

$$\frac{dx}{dt} = x^3(x-1)e^x.$$

(a) Explain why for any $x_0 \in (-\infty, \infty)$ there exists a unique solution to the equation satisfying the initial condition $x(0) = x_0$, at least on some time interval containing 0.

(b) Sketch the function $f(x) = x^3(x-1)e^x$ and identify the equilibrium points.

(c) Draw a phase diagram and identify the stable and unstable points.

(d) Sketch the equilibrium solutions in the tx plane. These equilibrium solutions divide the tx plane into regions. Sketch at least one solution curve in each of these regions.

(e) For the particular solution with initial condition $x(0) = -3$, what is the limit $\lim_{t\to+\infty} x(t)$?

at $\frac{dx}{dt} = f(t,x) = x^3(x-1)e^x$

f is continuous on $x \in (-\infty,\infty)$ and differentiable wrt x.

$\frac{\partial f}{\partial x} = 3x^2(x-1)e^x + x^3e^x + x^3(x-1)e^x$ is also continuous on the whole real line.

⇒ Picard's theorem of existence and uniqueness applies b/c its hypotheses are satisfied.

⇒ for any $(t_0, x_0) \in (-\infty,\infty)\times(-\infty,\infty)$ there exists a unique solution of f passing through that point.

⇒ There exists a unique solution for the equation satisfying the initial $x(0) = x_0$,