33B Visan

Midterm 1

October 17th

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Section: 1B w/ J. Yu =
- 1A Tuesda
- 1B Thursd
- 1C Tuesda
- 1D Thursd
- 1E Tuesda
- 1F Thursday with A. Lin

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Rules:

- There are **FOUR** problems for a total of 50 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	Σ
12	10	7	17	46

(1) (12 points)

A tank of capacity 1000 gallons initially contains 200 gallons of water in which 40 lbs of sugar are dissolved. Water containing 1 lb of sugar per gallon is pumped into the tank at the rate of 4 gallons per minute. The well-mixed solution is pumped out of the tank at the rate of 2 gallons per minute.

(a) When will the tank be full?

(b) Find the quantity of sugar (measured in lbs) in the tank at any time.

(c) What is the concentration of sugar in the tank at the time the tank is full?

12

a) $1000 = 200 + 4t - 2t \rightarrow 1000 = 200 + 2t \quad 800 = 2t$

$t = 400 \rightarrow \boxed{\text{After 400 minutes, it will be full}}$

b) Let x be the amount of sugar dissolved (in lbs)

$\frac{dx}{dt} = \text{rate in} - \text{rate out} \quad \left\{ \text{rate in} = \frac{1\,lb}{gal} \cdot \frac{4\,gal}{min} = \frac{4\,lb}{min} \right.$

$\text{rate out} = \frac{2\,gal}{min} \cdot \frac{x\,lb}{200+2t\ min}$

$\frac{dx}{dt} = 4 - \frac{2x}{200+2t} \rightarrow$ integrating factor $\frac{dx}{dt} = -\frac{2x}{200+2t}$

$x = e^{-\int -\frac{2}{200+2t}dt} = \quad u = 200+2t, \ du = 2t \quad \rightarrow e^{\int \frac{du}{u}} = e^{\ln|u|}$

$x(0) = 40\,lbs$

$= |u| = 200+2t$

$(200+2t)\int x' = \int 4(200+2t)dt = \int 800 + 8t\,dt = 800t + 4t^2 + C$

$x(t) = \frac{800t + 4t^2 + C}{200+2t} \rightarrow x(0) = 40$, so

$40 = \frac{0+0+C}{200} \quad C = 8000 \quad \boxed{x(t) = \frac{800t + 4t^2 + 8000}{200+2t}}$

c) Tank is full @ $t = 400 \rightarrow x = \frac{800(400) + 4(400)^2 + 8000}{1000} = \frac{320000 + 640000 + 8000}{1000}$

$400 \cdot 400 = 160000$

$= \frac{968{,}000}{1000} = 968\ lbs \rightarrow \frac{968}{1000} \leftarrow v(t) = \boxed{96.8\%\ \text{sugar}}$

(2) (12 points)

10

(a) Show that the differential equation

$$(4y + 2x - 2)dx + (6y + 4x - 1)dy = 0$$

is exact on the whole plane $(-\infty, \infty) \times (-\infty, \infty)$.

(b) Find the *explicit* solution to this equation subject to the initial condition $y(0) = 1$.

(c) What is the interval of existence of the solution you found in part (b)?

a) $\overbrace{(4y + 2x - 2)}^{P}\,dx + \overbrace{(6y + 4x - 1)}^{Q}\,dy = 0$

+2 - To be exact, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \frac{\partial}{\partial y}(4y + 2x - 2) = 4$

$\partial/\partial x\,(6y + 4x - 1) = 4$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the differential equation is exact. Furthermore, since Q and P, and the partial differentials are continuous and differentiable ok - you want to say this along P, Q on $(-\infty,\infty)$ the whole plane $\times(-\infty,\infty)$, we know that it is exact on this plane.

b) $F(x,y) = \int 4y + 2x - 2\, dx = 4xy + x^2 - 2x + \Phi(y)$

$\frac{\partial F}{\partial y} = Q \rightarrow \frac{\partial F}{\partial y} = \cancel{4x} + \Phi'(y) = 6y + \cancel{4x} - 1$

+7

$\Phi'(y) = 6y - 1 \qquad \Phi(y) = \int 6y - 1\, dy = 3y^2 - y + C$ for some $C \in (-\infty, \infty)$

$\hookrightarrow F(x,y) = 4xy + x^2 - 2x + 3y^2 - y = C$

@ $y(0) = 1 \rightarrow 4(0)(1) + 0 - 2(0) + 3 - 1 = C$

$C = 2$

$\boxed{F(x,y) = 4xy + x^2 - 2x + 3y^2 + y = 2}$

c) $3y^2 + 4xy - y(x^2 - 2x - 4) = 0 \rightarrow \frac{-(4x-1) \pm \sqrt{(4x-1)^2 - 12(x^2 - 2x - 2)}}{6}$

$y(4x - 1)$

$x = 1/4$ and $(x-1)(x-1) \rightarrow$

-2

The interval of existence must contain the initial

value $y(0)$ → therefore, the interval of existence is

$16x^2 - 8x + 1 = 12x^2 + 24x + 24$

$4x^2 + 16x + 25$ +1

$\Rightarrow$

$(\quad , \quad)$

-0

(3) (8 points)

Solve the initial-value problem

$$\frac{dx}{dt} = 2x - te^{2t} \quad \text{with} \quad x(0) = -2.$$

What is the interval of existence of the solution?

$$\frac{dx}{dt} = 2x - te^{2t} \quad w/ \; x(0) = -2$$

Using integrating factor → $\mu = e^{-\int 2\,dt} = e^{-2t}$

$e^{-2t} \cdot x' = \cancel{-e^{-2t}} \cdot t\cancel{e^{2t}}$ $\quad e^{-4}\int x' = \int -t\,dt \Rightarrow x = \left(-\frac{t^2}{2} + C\right)e^{2t}$

for some $C \in (-\infty, \infty)$

@ $x(0) = -2 \rightarrow \quad -2 = \left(\frac{0}{2} + C\right) \cdot 1 \quad C = -2$

$$\boxed{x(t) = \left(-\frac{t^2}{2} - 2\right)e^{2t}}$$

-1

The interval of of existence for this solution is the entire number line (all real numbers, $\mathbb{R}$). This is true because no value of t would make this solution discontinuous since t^2 is continuous for any t, and e^{2t} is also continuous for any value of t.

$\therefore I = -\infty < t < \infty$. ✓

7

bache

(4) (18 points)
Consider the differential equation

$$\frac{dx}{dt} = x^3(x-1)e^x. \quad = (x^4 - x^3)e^x = x^4e^x - x^3e^x$$

(a) Explain why for any $x_0 \in (-\infty, \infty)$ there exists a unique solution to the equation satisfying the initial condition $x(0) = x_0$, at least on some time interval containing 0.
(b) Sketch the function $f(x) = x^3(x-1)e^x$ and identify the equilibrium points.
(c) Draw a phase diagram and identify the stable and unstable points.
(d) Sketch the equilibrium solutions in the tx plane. These equilibrium solutions divide the tx plane into regions. Sketch at least one solution curve in each of these regions.
(e) For the particular solution with initial condition $x(0) = -3$, what is the limit $\lim_{t\to\infty} x(t)$?

a) For determining uniqueness, we need to use Picard's theorem →

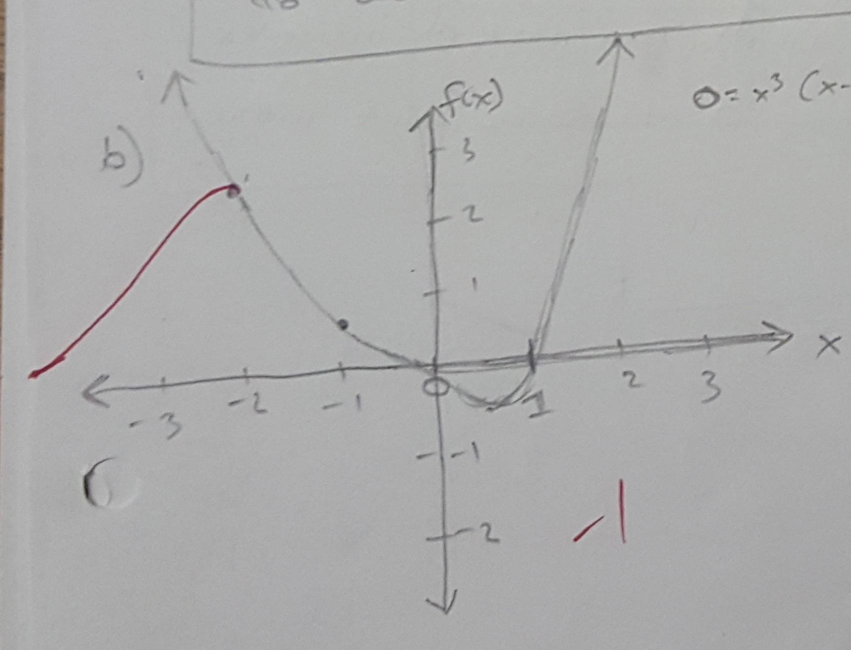
Let $F(x) = x^3(x-1)e^x$

$$\frac{\partial}{\partial x}(x^3)(x-1)(e^x) = \frac{\partial}{\partial x}(x^4e^x - x^3e^x) = 4x^3e^x + x^4e^x - (3x^2e^x + x^3e^x)$$

Since $F(x)$ is continuous and differentiable on the entire plane, and its partial derivative is also continuous on the entire plane, Picard's theorem tells us that if there is some equation that satisfies the initial condition $x(0) = x_0$, then it will be the only (and unique) solution to do so.

2

b)



$0 = x^3(x-1)e^x \rightarrow x=1, x=0, e^x = 0 \rightarrow$ not possible

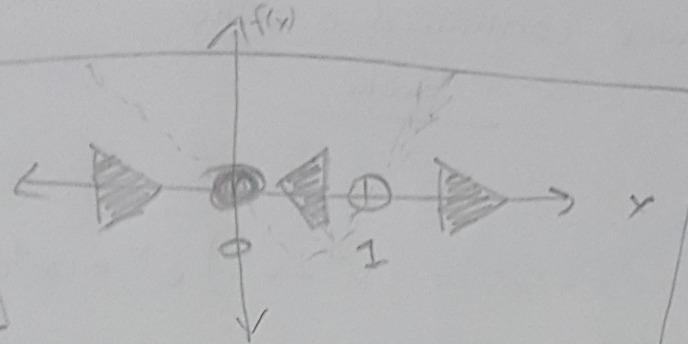
$x<0$ $(-1)(-2)\left(\frac{1}{e}\right)$
$(-8)(-3)\left(\frac{1}{e^2}\right)$

$0<x<1$ $0.53(-0.5)e^{1/2}$

$x>1$ $8 \cdot 1 \cdot e^2$

-1

c)



The stable point is at (0,0) and the unstable point is at (1,0).

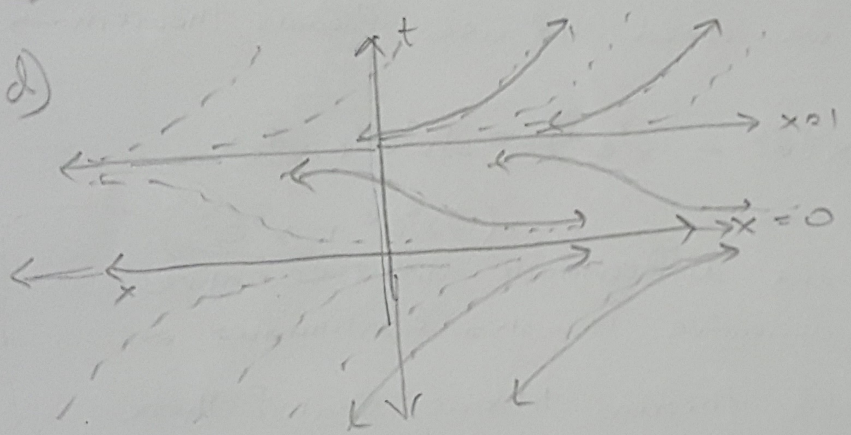
using the graph from part b,

$x > 1$ = increasing (above 0)

$0 < x < 1$ = below 0

$x < 0$ = above 0

d)



e) when $x(0) = -3$, $\lim_{t \to \infty} x(t) = 0$. The region for $x < 0$ in the tx graph from part d indicates that the graph will reach 0 at some point in the future (ie. $t \to \infty$).