Math33B Exam #2 Winter 2015

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By signing here, you confirm you are the person identified above and that all the work herein is solely your own.

Instructions:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this midterm exam.

Problem	Points	Score
1	12	12
2	12	10
3	12	12
4	12	5
5	12	12
Total	60	51

1. (12 points) Consider the first order ODE ,

$$\left(\frac{1}{1-x^2}\right)\frac{dx}{dt} = x. \quad (1)$$

(a) (4 points) Find all equilibrium solutions to (1), and for each solution determine whether it is stable or not.

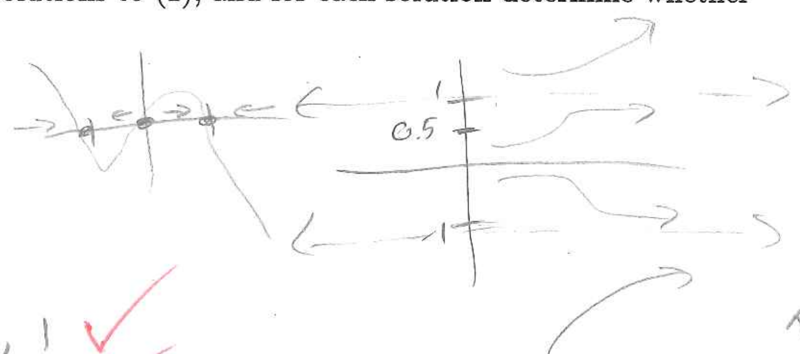
$x' = x \; \underline{1-x^2}$

$x' = x(1-x)(1+x)$

equi. soln $\Rightarrow$ $x = 0, 1, -1$

stable points : $-1, 1$

unstable point : 0



(b) (4 points) For the solution, $x(t)$, to initial value problem defined by the initial condition $x(0) = 0.5$, determine $\lim_{t\to\infty} x(t)$.

since the graph looks something like that,

$\lim_{t\to\infty} x(t)$ when $x(0) = 0.5$ will go to 1

(c) (4 points) Now find an expression for the general solution to (1).
[Hint: $\frac{1}{x(1-x^2)} = \frac{1}{x} - \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{2}\frac{1}{(x-1)}$.]

$\left(\frac{1}{1-x^2}\right)\cdot\frac{1}{x}\,dx = 1\,dt$

$\int\left(\frac{1}{x} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)}\right)dx = 1\,dt$

$\int\frac{1}{x}dx - \int\frac{1}{2(x+1)}dx - \int\frac{1}{2(x-1)}dx = \int 1\,dt$

$\ln(x) - \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) = t + c$

$\ln(x) - \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) = t+c$

$x + \left(\frac{x+1}{x-1}\right)^{-\frac{1}{2}} = e^{t+c}$

2. (12 points) Find the general solutions to

-6 2

(a) (4 points)

$$x'' - 7x' - 12x = 0$$

$\lambda^2 - 7\lambda - 12 = 0$

$\dfrac{7 \pm \sqrt{49+48}}{2}$ $\quad \dfrac{7 \pm \sqrt{97}}{2}$ $\quad \underline{C_1 e^{\frac{7+\sqrt{97}}{2}} + C_2 e^{\frac{7-\sqrt{97}}{2}} = y(t)}$

(b) (4 points)

$$x'' - 4x' + 4x = 0$$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$ $\quad \underline{(C_1 + C_2 t)e^{2t} = y(t)}$ ✓

$\lambda = 2$

e^{2t}

(c) (4 points)

$$x'' + 4x' + 8x = 0$$

$\lambda^2 - 4\lambda + 8 = 0$

$\dfrac{4 \pm \sqrt{16-32}}{2} = \dfrac{4 \pm 4i}{2} = 2 \pm 2i$ $\quad -2$

$e^{-2t}(A_1 \cos 2t + A_2 \sin 2t) = y(t)$

3. (12 points) For each of the following differential equations find the general solution.

(a) (4 points)

$$y'' + 9y = \sin 2t$$

$y(t) = C_1\cos 3t + C_2 \sin 3t + \frac{1}{5}\sin 2t$

$y(t) = y_h + y_p$

$y_h)$ $\lambda^2 + 9$

$\lambda = \pm 3i$

$C_1 \cos 3t + C_2 \sin 3t$

$y_p)$ $a\cos 2t + b\sin 2t = y_p(t)$

$-2a\sin 2t + 2b\cos 2t = y_p'(t)$

$-4a\cos 2t - 4b\sin 2t = y_p''(t)$

$-4(a\cos 2t + b\sin 2t) + 9(a\cos 2t + b\sin 2t) = \sin 2t$

$5a\cos 2t + 5b\sin 2t = \sin 2t$

$5a = 0$ $a = 0$ $5b = 1$ $b = \frac{1}{5}$ $\frac{1}{5}\sin 2t = y_p(t)$

(b) (4 points)

$$y'' + 3y' + 2y = 4e^{-3t}$$

$y(t) = y_h + y_p$

$y_h)$ $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -2, -1$

$C_1 e^{-2t} + C_2 e^{-t} = y_h$

$y_p)$ $ae^{-3t} = y_p$

$-3ae^{-3t} = y_p'$

$9ae^{-3t} = y_p''$

$9ae^{-3t} - \cancel{9ae^{-3t}} + 2ae^{-3t} = 4e^{-3t}$

$2a = 4$

$a = 2$

$2e^{-3t} = y_p$

$y(t) = C_1 e^{-2t} + C_2 e^{-t} + 2e^{-3t}$

(c) (4 points)

$$y'' + 9y = \sin 3t$$

$y(t) = y_h + y_p$

$y_h)$ $\lambda^2 - 9$

$\lambda = \pm 3i$

$C_1 \cos 3t + C_2 \sin 3t$

$\left.\begin{array}{l} y_p)\ a\cos 3t + b\sin 3t = y_p \\ -3a\sin 3t + 3b\cos 3t = y_p' \\ -9a\sin 3t - 9b\cos 3t = y_p'' \end{array}\right\}$ wrong

$\cancel{-9(a\sin 3t + b\cos 3t) + 9(a\cos 3t + b\sin 3t)} = \sin 3t$

$t(a\cos 3t + b\sin 3t) = y_p$

$(a\cos 3t + b\sin 3t) + t(-3a\sin 3t + 3b\cos 3t) = y_p'$

$(-3a\sin 3t + 3b\cos 3t) + (-3a\sin 3t + 3b\cos 3t) + t(-9a\sin 3t - 9b\cos 3t) = y_p''$

cont. in back

$y'' + 9y$

$(-6a\sin 3t + 6b\cos 3t) + t\cancel{(-9a\cos 3t - 9b\sin 3t)} + \cancel{9t(a\cos 3t + b\sin 3t)} = \sin 3t$

$-6a\sin 3t + 6b\cos 3t = \sin 3t$

$-6a = 1 \qquad 6b = 0$

$a = \frac{-1}{6} \qquad b = 0$

$y_p = t\left(\frac{-1}{6}\cos 3t\right)$

$y(t) = C_1\cos 3t + C_2\sin 3t + t\left(\frac{-1}{6}\cos 3t\right)$

$= \left(C_1 - \frac{t}{6}\right)\cos 3t + C_2\sin 3t$

4. (12 points) Given the differential equation,

$$(t^2+1)x'' - 2tx' + 2x = 6(t^2+1)^2, \tag{2}$$

with homogeneous solution,

$$x_h(t) = C_1 t + C_2(t^2-1), \tag{3}$$

find the general solution to the non-homogeneous equation (2).

$x_p = v_1 t + v_2 (t^2-1)$

Call this A

$$\begin{bmatrix} t & t^2-1 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 6(t^2+1)^{2} \end{bmatrix}$$

$\det(A) = 2t^2 - t^2 + 1$
t^2+1

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} 2t & 1-t^2 \\ -1 & t \end{bmatrix} \begin{bmatrix} 0 \\ 6(t^2+1)^2 \end{bmatrix}$$

5/12,

$$= \begin{bmatrix} 2t(t^2+1) & (1-t^2)(t^2+1) \\ -(t^2+1) & t(t^2+1) \end{bmatrix} \begin{bmatrix} 0 \\ 6(t^2+1)^2 \end{bmatrix}$$

$$= \begin{bmatrix} (1-t^2)(t^2+1)\,6(t^2+1)^2 \\ 6t(t^2+1)^3 \end{bmatrix}$$

$v_1' = 6(1-t^2)(t^2+1)^3$ v_1

$v_2' = 6t(t^2+1)^3$

48
63
111

$-6(t^4-1)(t^2+1)^2$

$(t^4+2t^2+1)(t^4-1)$

$(t^8+2t^6+\cancel{t^4}-\cancel{t^4}-2t^2-1)6$

$6t^8+12t^6-12t^2-6 = v_1'$

$v_1 = \frac{2}{3}t^9 + \frac{12}{7}t^7 - 4t^3 - 6t$

$6t(t^4+2t^2+1)(t^2+1)$

$6t(t^6+2t^4+t^2+t^4+2t^2+1$

$6t(t^6+3t^4+3t^2+1)$

$6t^7+18t^5+18t^3+6t = v_2'$

$v_2 = (\frac{3}{4}t^8 + 3t^6 + \frac{9}{2}t^4 + 3t^2)$ t^2-1

$(\frac{3}{4}t^{10} + 3t^8 + \frac{9}{2}t^6 + 3t^4 - \frac{3}{4}t^8 - 3t^6 - \frac{9}{2}t^4 - 3t^2$

$x_p = (\frac{2}{3}t^{10} + \frac{12}{7}t^8 - 4t^4 - 6t^2) + (\frac{3}{4}t^{10} + \frac{9}{4}t^8 + \frac{3}{2}t^6 - \frac{3}{2}t^4 - 3t^2)$

$= \frac{17}{12}t^{10} + \frac{111}{28}t^8 + \frac{3}{2}t^6 - \frac{11}{2}t^4 - 9t^2$

cont. in back

$$y(t) = c_1 t + c_2(t^2-1) + t^2\left(\frac{117}{12}t^8 + \frac{111}{28}t^6 + \frac{3}{2}t^4 - \frac{11}{2}t^2 - 9\right)$$

5. (12 points) Consider a spring-mass system used as the suspension on the wheels of a motorcycle. Assume that the mass of the motorcycle is 200 kg, the mass of a hypothetical rider is M kg (so that the total mass on the spring is $m = 200 + M$ kg), the spring constant is $k = 400$ kg/S^2, and the damping coefficient is $\mu = 800$ kg/S.

(a) (2 points) Write down an ODE describing this system in terms of m, μ and k.

$$\underbrace{(200+M)}_{m} y'' + \underbrace{(800)}_{\mu} y' + \underbrace{(400)}_{k} y = 0$$

(b) (6 points) What is the maximum mass, M, of a rider so that the rider never experiences oscillations?

over damped

$c > \omega_0$

$M = 200$ kg

$\frac{\mu}{m} = 2c \quad \omega = \sqrt{\frac{k}{m}}$

$\frac{\mu}{2(m)} = \sqrt{\frac{k}{m}}$

$\frac{800}{2(200+M)} = \sqrt{\frac{400}{200+M}}$

$\frac{800^2}{(2(200+M))^2} = \frac{400}{200+M}$

$\frac{200+M}{4(200+M)^2} = \frac{400}{800^2}$

$\frac{1}{4(200+M)} = \frac{1}{2 \cdot 800}$

$4(200+M) = 1600$

$800 + 4M = 1600$

$4M = 800$

$M = 200$

(c) (4 points) Find the general solution to the damped harmonic oscillation for the mass, M, you found in part (b).

$400y'' + 800y' + 400y = 0$

$y'' + 2y' + y = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0$

$\lambda = -1 \quad e^{-t}$

$y(t) = (C_1 + C_2 t)e^{-t}$