1. (12 points) Consider the first order ODE ,

$$\left(\frac{1}{1-x^2}\right)\frac{dx}{dt} = x. \tag{1}$$

(a) (4 points) Find all equilibrium solutions to (1), and for each solution determine whether it is stable or not.

$\frac{dx}{dt} = x - x^3$

$x' = x - x^3$

$x - x^3 = 0$

$x(1-x^2) = 0$

$x = 0, \pm 1$

let $f(x) = x^3$

$f'(x) = 1 - 3x^2$

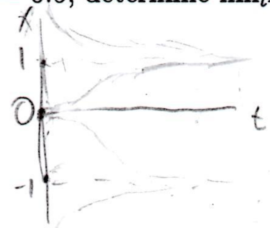
$f'(0) = 1 > 0$ $\boxed{\text{Unstable}}$

$f'(-1) = 1 - 3 = -2 < 0$ $\boxed{\text{Stable}}$

$f'(1) = 1 - 3 = -2 < 0$ $\boxed{\text{Stable}}$

(b) (4 points) For the solution, $x(t)$, to initial value problem defined by the initial condition $x(0) = 0.5$, determine $\lim_{t\to\infty} x(t)$.

phase plane



Between $x = 0$ and $x = 1$, $\lim_{t\to\infty} x(t)$ will approach $\boxed{x = 1}$

(c) (4 points) Now find an expression for the general solution to (1).
[Hint: $\frac{1}{x(1-x^2)} = \frac{1}{x} - \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{2}\frac{1}{(x-1)}$.]

$\frac{dx}{x(1-x^2)} = dt$

$\frac{1}{x(1-x^2)} = \frac{1}{x} - \frac{1}{2}\frac{1}{(x+1)} - \frac{1}{2}\frac{1}{(x-1)}$

$\int \left(\frac{1}{x} - \frac{1}{2}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x-1}\right)dx = t + C$

$\ln|x| - \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| = t + C$

$\ln|x| - \left(\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|\right) = t + C$

$\ln|x| - \ln\left|\left(\frac{x+1}{x-1}\right)^{1/2}\right| = t + C$

$\ln\left|x\left(\frac{x-1}{x+1}\right)^{1/2}\right| = t + C$

$\boxed{x\left(\frac{x-1}{x+1}\right)^{1/2} = Ae^{t}}$

2. (12 points) Find the general solutions to

(a) (4 points)

$$x'' - 7x' - 12x = 0$$

$\lambda^2 - 7\lambda - 12 = 0$

$\lambda = \frac{7 \pm \sqrt{49+48}}{2} = \frac{7 \pm \sqrt{97}}{2}$

$\boxed{y = C_1 e^{\frac{7+\sqrt{97}}{2}t} + C_2 e^{\frac{7-\sqrt{97}}{2}t}}$

(b) (4 points)

$$x'' - 4x' + 4x = 0$$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda = 2$

$y_1 = e^{2t}$ $y_2 = te^{2t}$

$\boxed{y = C_1 e^{2t} + C_2 te^{2t}}$

✓

(c) (4 points)

$$x'' + 4x' + 8x = 0$$

$\lambda^2 + 4\lambda + 8 = 0$

$\lambda = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i$

$y_1 = e^{-2t}(\cos 2t)$

$y_2 = e^{-2t}(\sin 2t)$

$\boxed{y = e^{-2t}(C_1 \cos 2t + C_2 \sin 2t)}$

3. (12 points) For each of the following differential equations find the general solution.

(a) (4 points)

$$y'' + 9y = \sin 2t$$

$y = y_h + y_p$

y_h: $\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$y_h = C_1 \cos 3t + C_2 \sin 3t$

$y_p = \frac{1}{5}\sin 2t$

$\boxed{y = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{5}\sin 2t}$

$y_p = a\cos 2t + b\sin 2t$

$y_p' = -2a\sin 2t + 2b\cos 2t$

$y_p'' = -4a\cos 2t - 4b\sin 2t$

$y'' + 9y = -4a\cos 2t - 4b\sin 2t + 9a\cos 2t + 9b\sin 2t$

$= 5a\cos 2t + 5b\sin 2t = \sin 2t$

$5a = 0 \quad a = 0$

$5b = 1 \quad b = \frac{1}{5}$

(b) (4 points)

$$y'' + 3y' + 2y = 4e^{-3t}$$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 1)(\lambda + 2) = 0$

$\lambda = -1, -2$

$y_h = C_1 e^{-t} + C_2 e^{-2t}$

$y = y_h + y_p$

$= \boxed{C_1 e^{-t} + C_2 e^{-2t} + 2e^{-3t}}$

$y_p = ae^{-3t}$ $\quad y_p = 2e^{-3t}$

$y_p' = -3ae^{-3t}$

$y_p'' = 9ae^{-3t}$

$\cancel{9ae^{-3t}} - \cancel{9ae^{-3t}} + 2ae^{-3t} = 4e^{-3t}$

$2a = 4$

$a = 2$

(c) (4 points)

$$y'' + 9y = \underline{\sin 3t}$$

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$y_h = C_1 \cos 3t + C_2 \sin 3t$

$y = C_1 \cos 3t + C_2 \sin 3t - \frac{1}{6} t\cos 3t$

$\sin 3t$ is homogeneous solution

$y_p = at\sin 3t + bt\cos 3t$

$y_p' = a\sin 3t + b\cos 3t + 3at\cos 3t - 3bt\sin 3t$

$y_p'' = 6a\cos 3t - \cancel{6b}\sin 3t - 9at\sin 3t - 9bt\cos 3t$

$6a\cos 3t - 6b\sin 3t = \sin 3t$

$a = 0 \quad -6b = 1$

$b = -\frac{1}{6}$

$y_p = -\frac{1}{6} t\cos 3t$

4. (12 points) Given the differential equation,

$$(t^2+1)x'' - 2tx' + 2x = 6(t^2+1)^2, \tag{2}$$

with homogeneous solution,

$$x_h(t) = C_1 t + C_2(t^2-1), \tag{3}$$

find the general solution to the non-homogeneous equation (2).

$x_p = v_1 t + v_2(t^2-1)$

$x_p' = v_1' t + v_2'(t^2-1) + v_1 + 2t v_2$

$v_1' t + v_2'(t^2-1) = 0$

$x_p'' = v_1' + 2t v_2' + 2v_2$

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$(t^2+1)[v_1' + 2t v_2' + 2v_2] - 2t(v_1 + 2t v_2) + 2v_1 t + 2v_2(t^2-1) = 6(t^2+1)^2$

$v_1'(t^2+1) + 2t(t^2+1)v_2' + \cancel{2t^2 v_2} + \cancel{2v_2} - \cancel{2tv_1} - \cancel{4t^2 v_2} + \cancel{2tv_1} + \cancel{2t^2 v_2} - \cancel{2v_2} = 6(t^2+1)^2$

$v_1' + 2t v_2' = 6(t^2+1) \qquad v_1' = 6t^2 + 6 - 2t v_2'$

$v_1' t + v_2'(t^2-1) = 0 \qquad 6t^3 + 6t - 2t^2 v_2' + v_2' t^2 - v_2' = 0$

$-t^2 v_2' - v_2' = -6(t^3+t)$

$v_2'(t^2+1) = 6t(t^2+1)$

$v_2' = 6t$

$v_2 = 3t^2$

$v_1' = 6t^2 + 6 - 2t(6t) = -6t^2 + 6$

$v_1 = -2t^3 + 6t$ ✓

$x_p = v_1 t + v_2(t^2-1)$

$= -2t^4 + 6t^2 + 3t^4 - 3t^2 =$

$t^4 + 3t^2$

$x = x_h + x_p = \boxed{C_1 t + C_2(t^2-1) + t^4 + 3t^2}$

5. (12 points) Consider a spring-mass system used as the suspension on the wheels of a motorcycle. Assume that the mass of the motorcycle is 200 kg, the mass of a hypothetical rider is M kg (so that the total mass on the spring is $m = 200 + M$ kg), the spring constant is $k = 400$ kg/S^2, and the damping coefficient is $\mu = 800$ kg/S.

(a) (2 points) Write down an ODE describing this system in terms of m, μ and k.

$f(t) = my'' + \mu y' + k$

(b) (6 points) What is the maximum mass, M, of a rider so that the rider never experiences oscillations?

$m\lambda^2 + \mu\lambda + k = 0$

$\lambda = \dfrac{-\mu \pm \sqrt{\mu^2 - 4km}}{2m}$

For critical damping, $\mu^2 = 4km$

$200 + M = \dfrac{\mu^2}{4k}$

$M = \dfrac{800^2}{4 \cdot 400} - 200 = 400 - 200 = \boxed{200\,kg}$

(c) (4 points) Find the general solution to the damped harmonic oscillation for the mass, M, you found in part (b).

$m = 200 + M = 200 + 200 = 400$

$400y'' + 800y' + 400 = 0$

$y'' + 2y' + 1 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0$

$\lambda = -1$

$\boxed{y = C_1 e^{-t} + C_2 t e^{-t}}$