Math 33B Exam #1 Winter 2015

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By signing here, you confirm you are the person identified above and that all the work herein is solely your own.

Instructions:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this midterm exam.

Problem	Points	Score
1	10	10
2	10	9
3	10	7
4	10	10
5	10	7
Total	50	43

1. (10 points) Find the value of a for which the ODE

$$x^2y' + 3x^2 + axy = 0 \qquad (1)$$

is exact, and then find the general solution to the equation (1) using this value of a.

$$x^2 \frac{\partial y}{\partial x} + 3x^2 + axy = 0$$

$$(3x^2 + axy)\,\partial x + x^2\,\partial y = 0$$

$$x^3 + x^2y \qquad yx^2 + \cancel{f(x)}\, x^3$$

$$\boxed{a = 2}$$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy$$

$$\boxed{F(x,y) = x^2y + x^3}$$

10

2. (10 points) Given the initial value problem,

$$\frac{dx}{dt} + \frac{x}{t} = 2; \quad x(1) = 0 \tag{2}$$

5

(a) (5 points) Use an integrating factor to solve the initial value problem (2).

$$u(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t \checkmark$$

$$tx' + x = 2t$$

$$\int (tx)' = \int 2t$$

$$tx = t^2 + C$$

$$x = t + \frac{C}{t}$$

$$0 = 1 + \frac{C}{1} \checkmark \quad \boxed{-C = 1}$$

$$\boxed{x(t) = t - \frac{1}{t}}$$

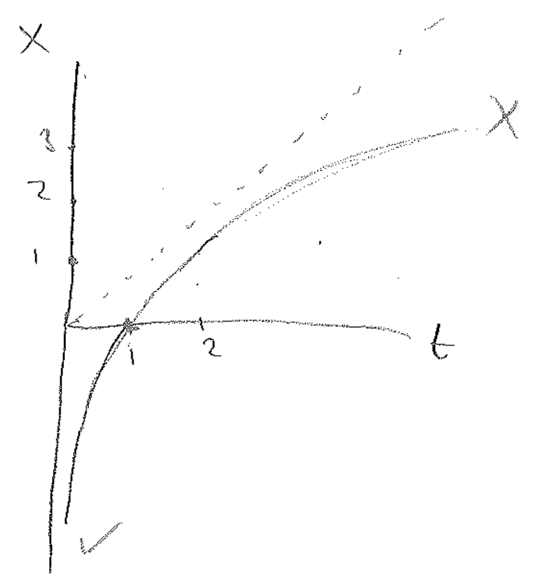
4

(b) (5 points) Determine the interval of existence and provide a sketch of the solution.

interval of existence $\cancel{(-\infty, 0)}$ or $\boxed{(0, \infty)}$ ✓

$.5 - \frac{1}{.5}$

$.5 < \frac{1}{\frac{1}{2}}$



✓

3. (10 points) A container which holds 200L is initially filled with 100L of pure water. A salt-water solution of 1kg salt per liter is pumped into the container at a rate of 3L per minute. The tank drains at a rate of 2L per minute.

(a) (5 points) Determine and solve the initial value problem which describes the mass in kilograms of salt dissolved in the container during the period of time before it completely fills with water.

$V = 200$ $\quad 1\,kg/L \cdot 3L/min = 3\,kg/min$

$x' = 3 - \frac{2L}{min}\frac{x\,kg}{100+t}$ $\quad V' = 3-2$, $V = t$

$x' + \frac{2}{100+t}x = 3$ $\quad \boxed{x' = 3 - \frac{2x}{100+t}}$

$e^{\int \frac{2}{100+t}\,dt} = e^{2\ln(100+t)} = (100+t)^2$

$\int ((100+t)^2 x)' = \int 3(100+t)^2$

$x(0)=0$

$(100+t)^2 x = \int 30{,}000 + 600t + 3t^2$

$\cancel{(100+t)^2}\, x = (100+t)^3 + C$ $\quad$ $x = 100+t + \frac{C}{(100+t)^2}$

$x = 100 + t$ $\quad$ $0 = 100 + \frac{C}{10000}$

$x(100) = 100 + 100 = 200$

(b) (5 points) At some time the container will be completely full and the excess salt-water in the container will overflow to keep the volume of water at 200 liters. Determine and solve the initial value problem for the mass of dissolved salt in the container after the container is completely full, assuming that the initial amount of salt in the container is the amount you found in part (a) at the time the container completely fills.

$x' = 1\,kg/L \cdot 3L/min - \frac{3L}{min}\frac{x}{200}$

$x' = 3 - \frac{3x}{200}$

$x' + \frac{3x}{200} = 3$ $\quad e^{\frac{3}{200}\int dt} = e^{\frac{3t}{200}}$

$\int (e^{\frac{3t}{200}}x)' = \int 3e^{\frac{3t}{200}}$

$e^{\frac{3t}{200}}x = 200e^{\frac{3t}{200}} + C$

$x = 200 + \frac{C}{e^{\frac{3t}{200}}}$

$200 = 200 + \frac{C}{e^0} = 0 = C$

$x(0) = 200$

$x = 200$

4

4. (10 points) Consider the differential equation

$$(2\cos 2x)dx + \left(\frac{e^y}{1+y^2} - \sin 2x\right) dy = 0$$

(a) (5 points) Find an integrating factor μ, which depends on only one variable, for this equation.

$$\mu'(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{2\cos 2x}\left(0 + (0 + 2\cos 2x)\right) = \frac{\cancel{2\cos 2x}}{\cancel{2\cos 2x}} = 1$$

$$\mu'(y) = \mu \quad \boxed{\mu = -e^{y}} \quad e^{-\int 1\,dy} = e^{-y}$$

✓

(b) (5 points) Solve this equation.

$$-2e^{-y}\cos 2x + \left(\frac{1}{1+y^2} - e^{-y}\sin 2x\right)dy = 0$$

$$F(x,y) = \int -2e^{y}\cos 2x \, \partial x$$

$$= e^{y}\sin 2x + \phi(y)$$

$$\frac{\partial F}{\partial y} = -e^{-y}\sin 2x + \phi'(y)$$

✓

$$\boxed{F(x,y) = \arctan(y) + e^{-y}\sin 2x}$$

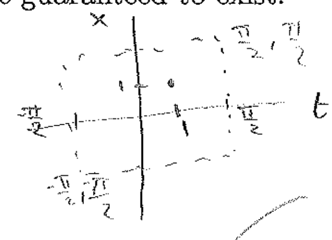
5. (10 points) Consider the initial value problem $x^{2/3}$ can be any number

$$\frac{dx}{dt} = \frac{x^{2/3}}{\cos t \cos x} \quad \text{with } x(0) = 0.$$

$t \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\quad x \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$

(a) (5 points) Precisely describe the largest rectangle R in the t, x-plane, which contains $(1, 1)$ and for which all initial value problems in R are guaranteed to exist.

$-\frac{\pi}{2} < t < \frac{\pi}{2} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$



(b) (5 points) What is the largest rectangle R, with $(1, 1) \in R$, such that the solutions in R are guaranteed to be unique?

$F(x,t) = \frac{x^{2/3}}{\cos t \cos x} \qquad F'(x,t) = \cos t \cos x \left(\frac{2}{3} x^{-1/3}\right) - x^{2/3}$

derive F to see if continuous on rectangle R

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