Course 33B – Midterm 2

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Name:

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Section:

Signatur

You have 50 minutes to complete this exam. Calculators and textbooks are not allowed. Show all work. Unsupported or illegible answers will receive no credit. There are a total of 100 points on this exam.

Problem	Score	Points
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

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1. [*20 points*] Use a first derivative test for stability to classify the equilibrium solutions of the following systems. If this test fails because $f'(y^*)=0$, use a graphical argument to decide the stability.

(a) $y'(t)=1-e^{y^3}$; (b) $y'(t)=e^{y}\cos(y)$.

$0 = 1 - e^{y^3}$

$e^{y^3} = 1$

$y^3 = 0$

$y^* = 0$

$0 = e^y \cos(y)$

$y^* = \frac{\pi}{2} + 2\pi n,\ \frac{3\pi}{2} + 2\pi n,\ n \in \mathbb{Z}$

$f'(y) = -e^{y^3} \cdot 3y$

$f'(0) = 0$

$f'(y) = e^y \cos(y) - e^y \sin(y)$

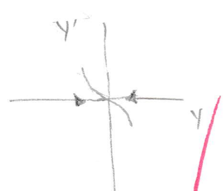
$f'(\frac{\pi}{2} + 2\pi n) = -e^{\pi/2 + 2\pi n}(1) < 0$

$f'(\frac{3\pi}{2} + 2\pi n) = -e^{3\pi/2 + 2\pi n}(-1) > 0$

If $dy > 0$, then $y'(dy) < 0$.

If $dy < 0$, then $y'(dy) > 0$.

So the picture looks like:



Equilibrium solution:

0 Stable

Equilibrium Solutions:

$\frac{\pi}{2} + 2\pi n$	Stable
$\frac{3\pi}{2} + 2\pi n$	unstable

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2. [*20 points*] Find the solution of the following initial value problem

$$y''(t)-2y'(t)-3y(t)=0, \qquad y(0)=0, \quad y'(0)=4.$$

a) Find the general solution; b) Find a particular solution.

a) $\lambda^2-2\lambda-3=0$ $\qquad$ * $y:=e^{\lambda t}$

$(\lambda+1)(\lambda-3)=0$

$\lambda=-1, 3$

$y=c_1e^{-t}+c_2e^{3t}$

b) $y'=-c_1e^{-t}+3c_2e^{3t}$

$$\begin{cases} y(0)=0=c_1+c_2 \\ y'(0)=4=-c_1+3c_2 \end{cases}$$

$4=4c_2 \qquad 0=c_1+1$

$c_2=1 \qquad c_1=-1$

$y=-e^{-t}+e^{3t}$

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3. [*20 points*] Find the general solution of the following equation

$$y''(t)+9y(t)=24\cos(3t)-6\sin(3t).$$

a) Find a solution of the homogeneous equation; b) Find a particular solution of the nonhomogeneous equation by the method of undetermined coefficients; c) Write the general solution of the ODE.

a) $\lambda^2+9=0$ $*\ y_h := e^{\lambda t}$

$\lambda^2=-9$

$\lambda=\pm 3i$

$y_h = e^{\pm 3it}$

$y_h = \text{cis}(\pm 3t)$

$y_h = c_1\cos(3t)+c_2\sin(3t)$ ✓

b) Let $y_p = c_1 t\cos(3t)+c_2 t\sin(3t)$ be a solution.

Then $y_p' = c_1(\cos(3t)-3t\sin(3t))+c_2(\sin(3t)+3t\cos(3t))$

$= (c_1+c_2 3t)\cos(3t)+(c_2-c_1 3t)\sin(3t)$

and $y_p'' = [3c_2\cos(3t)-3(c_1+c_2 3t)\sin(3t)]+[-3c_1\sin(3t)+3(c_2-c_1 3t)\cos(3t)]$

$= (6c_2-9c_1t)\cos(3t)+(-6c_1-9c_2t)\sin(3t)$

$y_p''+9y_p = 24\cos(3t)-6\sin(3t)$

$(6c_2-\cancel{9c_1t})\cos(3t)+(-6c_1-\cancel{9c_2t})\sin(3t)+\cancel{9c_1t\cos(3t)}+\cancel{9c_2t\sin(3t)} = 24\cos(3t)-6\sin(3t)$

$6c_2\cos(3t)-6c_1\sin(3t)=24\cos(3t)-6\sin(3t)$

$$\begin{cases}6c_2=24\\-6c_1=-6\end{cases} \Rightarrow \begin{cases}c_2=4\\c_1=1\end{cases} \Rightarrow y_p = t\cos(3t)+4t\sin(3t)$$ ✓

c) $y = y_h+y_p$

$y=[c_1\cos(3t)+c_2\sin(3t)]+[t\cos(3t)+4t\sin(3t)]$ ✓

4. [*20 points*] Use variation of parameters to find a particular solution to the following equation

$$y''(t) - 2y'(t) + y(t) = e^t \ln(t), \quad t > 0.$$

$\lambda^2 - 2\lambda + 1 = 0$ * $y_h = e^{\lambda t}$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

$y_h = c_1 e^t + c_2 t e^t$

$$W(t) = \det \begin{bmatrix} e^t & te^t \\ e^t & e^t + te^t \end{bmatrix}$$

$= e^t(e^t + te^t) - e^t(te^t)$

$= e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}}$

$= e^{2t}$

$y_p = v_1 e^t + v_2 t e^t$

$$v_1 = -\int \frac{y_2 \cdot f(t)}{W(t)} dt = -\int \frac{t\cancel{e^t} \cdot \cancel{e^t} \ln(t)}{\cancel{e^{2t}}} dt = -\int t \ln(t)\, dt$$

* $u = \ln(t)$ $dv = t\, dt$

$du = \frac{1}{t}\, dt$ $v = \frac{1}{2}t^2$

$= -\left[\ln(t) \cdot \frac{1}{2}t^2 - \int \frac{1}{2}t^{\cancel{2}} \cdot \frac{1}{\cancel{t}}\, dt\right]$

$= -\left[\frac{1}{2}t^2 \ln(t) - \frac{1}{2}\int t\, dt\right]$

$= -\left[\frac{1}{2}t^2 \ln(t) - \frac{1}{4}t^2\right] = \frac{1}{4}t^2 - \frac{1}{2}t^2 \ln(t)$

$$v_2 = \int \frac{y_1 \cdot f(t)}{W(t)} dt = \int \frac{\cancel{e^t} \cdot \cancel{e^t} \ln(t)}{\cancel{e^{2t}}} dt = \int \ln(t)\, dt$$

* $u = \ln(t)$ $dv = dt$

$du = \frac{1}{t}\, dt$ $v = t$

$= t\ln(t) - \int \cancel{t} \cdot \frac{1}{\cancel{t}}\, dt$

$= t\ln(t) - t$

$y_p = \left[\frac{1}{4}t^2 - \frac{1}{2}t^2 \ln(t)\right] e^t + \left[t\ln(t) - t\right] te^t$ ✓

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5. [*20 points*] For the following system, find the eigenvalues, eigenvectors, and the general solution, classify the equilibrium solution at the origin.

$$y'(t) = Ay, \text{ where } A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

$\vec{y}' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{y}$

$\tau = 1 + 1 = 2$

$\Delta_{curve} = \frac{1}{4}\tau^2 = 1$

$\Delta = 1 - 4 = -3$

$\tau^2 - 4\Delta = 4 + 12 = 16$

$\Delta < 0$, so the equilibrium point at the origin is a saddle.

$0 = \det(A - \lambda I)$

$0 = \det \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}$

$0 = (1-\lambda)^2 - 4$

$0 = (\lambda^2 - 2\lambda + 1) - 4$

$0 = \lambda^2 - 2\lambda - 3 \quad (\stackrel{\checkmark}{=} \lambda^2 - \tau\lambda + \Delta)$

$0 = (\lambda + 1)(\lambda - 3)$

$\lambda = -1, 3 \quad \leftarrow$ eigenvalues

$E_{\lambda_1} = \ker \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \text{span}\left\langle \begin{matrix} 2 \\ -1 \end{matrix} \right\rangle$

$E_{\lambda_2} = \ker \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \text{span}\left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle$

eigenvectors

$\vec{y} = c_1 e^{-t} \left\langle \begin{matrix} 2 \\ -1 \end{matrix} \right\rangle + c_2 e^{3t} \left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle$

$\uparrow$

general solution

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