Course 33B – Midterm 1

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You have 50 minutes to complete this exam. Calculators and textbooks are not allowed. Show all work. Unsupported or illegible answers will receive no credit. There are a total of 100 points on this exam.

Problem	Score	Points
1	18	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	98	100

18

1. [*20 points*] Find the exact solution of the following initial value problem and indicate the interval of existence

$$y' = (x-1)\tan(y), \qquad y(0) = \tfrac{\pi}{2}.$$

$\frac{dy}{dx} = (x-1)\tan(y)$

$\frac{1}{\tan(y)}\,dy = (x-1)\,dx$

$\int \frac{1}{\tan(y)}\,dy = \int (x-1)\,dx$

$\int \frac{\cos(y)}{\sin(y)}\,dy = \int (x-1)\,dx$

* $u := \sin(y)$ −2

$du = \cancel{-}\cos(y)\,dy$

✓ $\cos(y)\,dy = -du$

$\cancel{-}\int \frac{1}{u}\,du = \int (x-1)\,dx$

$\cancel{-}\ln|u| = \frac{1}{2}x^2 - x + c$

$\cancel{-}\ln|\sin(y)| = \frac{1}{2}x^2 - x \cancel{+c}$

* $\cancel{-}\ln|\sin(\frac{\pi}{2})| = 0 - 0 + c$

$\cancel{-}\ln|1| = c$

$c = 0$

$\ln|\sin(y)| = x - \frac{1}{2}x^2$ $\frac{1}{2}x^2 - x$

$|\sin(y)| = e^{x - \frac{1}{2}x^2}$ $e^{\frac{1}{2}x^2 - x}$

$\sin(y) = \oplus e^{x - \frac{1}{2}x^2}$ $\oplus e^{\frac{1}{2}x^2 - x}$

* take positive sign (based on initial condition)

$y = \sin^{-1}(e^{x - \frac{1}{2}x^2})$ $y = \sin^{-1}(e^{\frac{1}{2}x^2 - x})$

IOE: $\cancel{-1 \le} e^{x - \frac{1}{2}x^2} \le 1$

$x - \frac{1}{2}x^2 \le 0$

$x(x - \frac{1}{2}) \le 0$

\+ − +

0 $\frac{1}{2}$

~~$0 \le x \le \frac{1}{2}$~~ okay if y was correct

IOE: $\cancel{-1 \le} e^{\frac{1}{2}x^2 - x} \le 1$

$\frac{1}{2}x^2 - x \le 0$

$x(\frac{1}{2}x - 1) \le 0$

$x(x-2) \le 0$

\+ − +

0 2

$0 \le x \le 2$

20

2. [*20 points*] Use the method of the integrating factor to find the solution of the following initial value problem

$$(1+x)y' - y = (1+x)^4, \qquad y(0) = 1.$$

$(1+x)y' = y + (1+x)^4$

$y' = \underbrace{\frac{1}{1+x}}_{a} y + \underbrace{(1+x)^3}_{f}$

$* \quad u := e^{-\int a\,dx}$

$u = e^{-\int \frac{1}{1+x}\,dx}$

$u = e^{-\ln|1+x|}$

↓ $u = \frac{1}{1+x}$

$u\left(y' - \frac{1}{1+x}y\right) = u(1+x)^3$

$\frac{1}{1+x}y' - \frac{1}{(1+x)^2}y = (1+x)^2$

$\left(\frac{1}{1+x}y\right)' = (1+x)^2$

$\frac{1}{1+x}y = \int (1+x)^2\,dx$

$\frac{1}{1+x}y = \int (x^2+2x+1)\,dx$

$\frac{1}{1+x}y = \frac{1}{3}x^3 + x^2 + x + c$ $\qquad * \; \frac{1}{1}(1) = 0 + 0 + 0 + c$

$c = 1$

$\frac{1}{1+x}y = \frac{1}{3}x^3 + x^2 + x + 1$ ←

$y = (1+x)\left(\frac{1}{3}x^3 + x^2 + x + 1\right)$ ✓

20

3. [*20 points*] Use the variation of parameters technique to find the solution of the following initial value problem

$$y' - xy = e^{\frac{x^2}{2}}, \qquad y(0) = 4.$$

$$y' = \underbrace{x}_{a}y + \underbrace{e^{\frac{x^2}{2}}}_{f}$$

$$* \quad y_h = e^{\int a\,dx}$$

$$y_h = e^{\int x\,dx}$$

$$y_h = e^{\frac{1}{2}x^2}$$ ✓

↓

√ $y = vy_h$

$$(ve^{\frac{1}{2}x^2})' = x(ve^{\frac{1}{2}x^2}) + e^{\frac{x^2}{2}}$$

$$v' \cdot e^{\frac{1}{2}x^2} + \cancel{v \cdot xe^{\frac{1}{2}x^2}} = \cancel{x(ve^{\frac{1}{2}x^2})} + e^{\frac{x^2}{2}}$$

$$v' \cdot \cancel{e^{\frac{1}{2}x^2}} = \cancel{e^{\frac{x^2}{2}}} \quad 1$$

$$v' = 1$$

$$v = \int 1\,dx$$

$$v = x + c$$ ✓

↓

$$y = vy_h$$

$$y = (x+c)e^{\frac{1}{2}x^2}$$

$$y = (x+4)e^{\frac{1}{2}x^2}$$ ✓ ←

$$* \quad 4 = ce^{\cancel{0}}$$

$$c = 4$$

20

4. [*20 points*] Use the integrating factor depending on x only, to find the general solution of the following differential equation

$$\underbrace{(xy-2)}_{P}dx+\underbrace{(x^2-xy)}_{Q}dy=0.$$

$$\frac{\partial P}{\partial y} = x \qquad \frac{\partial Q}{\partial x} = 2x - y$$

$$* \; h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2 - xy}\left[x - (2x-y)\right] = \frac{1}{x\cancel{(x-y)}}\left[-\cancel{(x-y)}\right] = -\frac{1}{x} \; ✓$$

$$\mu(x) = e^{\int h(x)\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x} \; ✓$$

$$\frac{(xy-2)}{x}dx + \frac{(x^2-xy)}{x}dy = 0$$

$$\left(y - \frac{2}{x}\right)dx + (x-y)\,dy = 0$$

$$* \; \frac{\partial P_2}{\partial y} \overset{?}{=} \frac{\partial Q_2}{\partial x}$$

$$1 \overset{✓}{=} 1 \; ✓$$

The DE is now exact!

$$F(x,y) = \int\left(y - \frac{2}{x}\right)dx = \int (x-y)\,dy$$

$$F(x,y) = xy - 2\ln|x| + c(y) = xy - \frac{1}{2}y^2 + c(x)$$

$$\underset{\substack{\text{must equal} \\ -\frac{1}{2}y^2}}{\uparrow} \qquad \underset{\substack{\text{must equal} \\ -2\ln|x|}}{\uparrow}$$

✓

$$F(x,y) = xy - 2\ln|x| - \frac{1}{2}y^2 = c$$

$$-\frac{1}{2}y^2 + xy - 2\ln|x| - c = 0$$

$$y = \frac{-x \pm \sqrt{x^2 - 4\left(-\frac{1}{2}\right)\left(-2\ln|x| - c\right)}}{2\left(-\frac{1}{2}\right)}$$

20

5. [*20 points*] Find the general solution of the homogeneous equation

$$\underbrace{(x^2 - xy + y^2)dx}_{P} - \underbrace{x^2}_{Q.}dy = 0.$$

$P(tx,ty) = (tx)^2 - (tx)(ty) + (ty^2) = t^2(x^2 - xy + y^2) = t^2\ P(x,y)$

$Q(tx,ty) = -(tx)^2 = t^2(-x^2) = t^2\ Q(x,y)$

This is a homogeneous equation of degree 2.

✓

Let $y = xv$.

$[x^2 - x(xv) + (xv)^2]\,dx - x^2(dx\ v + x\ dv) = 0$

$\cancel{x^2}(1 - v + v^2)\,dx - \cancel{x^2}(dx\ v + x\ dv) = 0$

$(v^2 - v + 1)\,dx - (dx\ v + x\ dv) = 0$

$(v^2 - 2v + 1)\,dx - x\ dv = 0$ ✓

$(v^2 - 2v + 1)\,dx = x\ dv$

$\frac{1}{x}\,dx = \frac{1}{v^2 - 2v + 1}\,dv$

$\int \frac{1}{x}\,dx = \int \frac{1}{v^2 - 2v + 1}\,dv$ ✓

$\int \frac{1}{x}\,dx = \int \frac{1}{(v-1)^2}\,dv$

$\ln|x| = -\frac{1}{v-1} + C$ ✓

$\ln|x| = -\frac{1}{\frac{y}{x} - 1} + C$

$C - \ln|x| = \frac{1}{\frac{y}{x} - 1}$

$\frac{1}{C - \ln|x|} = \frac{y}{x} - 1$

$\frac{1}{C - \ln|x|} + 1 = \frac{y}{x}$ ✓

$y = \frac{x}{C - \ln|x|} + x$