

# Course 33B – Midterm 1

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April 22th, 2016

Name:

ID number:

Section:

Signature:



You have 50 minutes to complete this exam. Calculators and textbooks are not allowed. Show all work. Unsupported or illegible answers will receive no credit. There are a total of 100 points on this exam.

Problem	Score	Points
1	18	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	98	100

1. [20 points] Find the exact solution of the following initial value problem and indicate the interval of existence

$$y' = (x-1)\tan(y), \quad y(0) = \frac{\pi}{2}.$$

$$\frac{dy}{dx} = (x-1)\tan(y)$$

$$\frac{1}{\tan(y)} dy = (x-1) dx$$

$$\int \frac{1}{\tan(y)} dy = \int (x-1) dx$$

$$\int \frac{\cos(y)}{\sin(y)} dy = \int (x-1) dx$$

\*  $u = \sin(y)$  -2

$$du = \cos(y) dy$$

$$\int \cos(y) dy = \int -du$$

$$+ \int \frac{1}{u} du = \int (x-1) dx$$

$$+ \ln|u| = \frac{1}{2}x^2 - x + c$$

$$+ \ln|\sin(y)| = \frac{1}{2}x^2 - x + c$$

\*  $\ln|\sin(\frac{\pi}{2})| = 0 - 0 + c$

$$\ln|\sin(y)| = x - \frac{1}{2}x^2 \quad \frac{1}{2}x^2 - x$$

$$+ \ln|1| = c$$

$$|\sin(y)| = e^{x - \frac{1}{2}x^2} \quad e^{\frac{1}{2}x^2 - x}$$

$$c = 0$$

$$\sin(y) = \oplus e^{x - \frac{1}{2}x^2} \quad \oplus e^{\frac{1}{2}x^2 - x}$$

\* take positive sign  
(based on initial condition)

$$y = \sin^{-1}(e^{x - \frac{1}{2}x^2})$$

$$y = \sin^{-1}(e^{\frac{1}{2}x^2 - x})$$

IOE:  ~~$x \leq e^{x - \frac{1}{2}x^2} \leq 1$~~

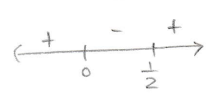
IOE:  ~~$x \leq e^{\frac{1}{2}x^2 - x} \leq 1$~~

$$x - \frac{1}{2}x^2 \leq 0$$

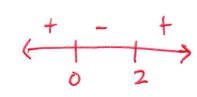
$$\frac{1}{2}x^2 - x \leq 0$$

$$x(x - \frac{1}{2}) \leq 0$$

$$x(\frac{1}{2}x - 1) \leq 0$$



$$x(x - 2) \leq 0$$



$$0 \leq x \leq 2$$

~~the answer is  $x \leq \frac{1}{2}$~~  okay if y was correct

20

2. [20 points] Use the method of the integrating factor to find the solution of the following initial value problem

$$(1+x)y' - y = (1+x)^4, \quad y(0) = 1.$$

$$(1+x)y' = y + (1+x)^4$$

$$y' = \underbrace{\frac{1}{1+x}}_a y + \underbrace{(1+x)^3}_f$$

$$* u := e^{-\int a dx}$$

$$u = e^{-\int \frac{1}{1+x} dx}$$

$$u = e^{-\ln|1+x|}$$

$$\checkmark u = \frac{1}{1+x}$$

$$u(y' - \frac{1}{1+x}y) = u(1+x)^3$$

$$\frac{1}{1+x}y' - \frac{1}{(1+x)^2}y = (1+x)^2$$

$$\left(\frac{1}{1+x}y\right)' = (1+x)^2$$

$$\frac{1}{1+x}y = \int (1+x)^2 dx$$

$$\frac{1}{1+x}y = \int (x^2 + 2x + 1) dx$$

$$\frac{1}{1+x}y = \frac{1}{3}x^3 + x^2 + x + c$$

$$* \frac{1}{1}(1) = 0 + 0 + 0 + c$$

$$\frac{1}{1+x}y = \frac{1}{3}x^3 + x^2 + x + 1$$

$$c = 1$$

$$y = (1+x)\left(\frac{1}{3}x^3 + x^2 + x + 1\right)$$



20

3. [20 points] Use the variation of parameters technique to find the solution of the following initial value problem

$$y' - xy = e^{\frac{x^2}{2}}, \quad y(0) = 4.$$

$$y' = \underbrace{xy}_a + \underbrace{e^{\frac{x^2}{2}}}_f$$

$$* y_h = e^{\int a \, dx}$$

$$y_h = e^{\int x \, dx}$$

$$y_h = e^{\frac{1}{2}x^2}$$

↓

$$\checkmark y = v y_h$$

$$(v e^{\frac{1}{2}x^2})' = x (v e^{\frac{1}{2}x^2}) + e^{\frac{x^2}{2}}$$

$$v' \cdot e^{\frac{1}{2}x^2} + v \cdot x e^{\frac{1}{2}x^2} = x (v e^{\frac{1}{2}x^2}) + e^{\frac{x^2}{2}}$$

$$v' \cdot e^{\frac{1}{2}x^2} = e^{\frac{x^2}{2}}$$

$$v' = 1$$

$$v = \int 1 \, dx$$

$$v = x + c$$

↓

$$y = v y_h$$

$$y = (x+c) e^{\frac{1}{2}x^2}$$

$$y = (x+4) e^{\frac{1}{2}x^2}$$

$$* 4 = c e^0$$

$$c = 4$$

4. [20 points] Use the integrating factor depending on  $x$  only, to find the general solution of the following differential equation

$$\underbrace{(xy-2)}_P dx + \underbrace{(x^2-xy)}_Q dy = 0.$$

$$\frac{\partial P}{\partial y} = x \quad \frac{\partial Q}{\partial x} = 2x - y$$

$$* h(x) = \frac{1}{Q} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{1}{x^2 - xy} [x - (2x - y)] = \frac{1}{x(x-y)} [-\cancel{(x-y)}] = -\frac{1}{x} \quad \checkmark$$

$$\downarrow \mu(x) = e^{\int h(x) dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{x} \quad \checkmark$$

$$\frac{(xy-2)}{x} dx + \frac{(x^2-xy)}{x} dy = 0$$

$$\left(y - \frac{2}{x}\right) dx + (x-y) dy = 0$$

$$* \frac{\partial P_2}{\partial y} \stackrel{?}{=} \frac{\partial Q_2}{\partial x}$$

$$1 \stackrel{\checkmark}{=} 1 \quad \checkmark$$

The DE is now exact!

$$F(x,y) = \int \left(y - \frac{2}{x}\right) dx = \int (x-y) dy$$

$$F(x,y) = xy - 2 \ln|x| + c(y) = xy - \frac{1}{2}y^2 + c(x)$$

$$\begin{array}{c} \uparrow \\ \text{must equal} \\ -\frac{1}{2}y^2 \end{array} \quad \checkmark$$

$$\begin{array}{c} \uparrow \\ \text{must equal} \\ -2 \ln|x| \end{array}$$

$$F(x,y) = xy - 2 \ln|x| - \frac{1}{2}y^2 = c$$

$$-\frac{1}{2}y^2 + xy - 2 \ln|x| - c = 0$$

$$y = \frac{-x \pm \sqrt{x^2 - 4(-\frac{1}{2})(-2 \ln|x| - c)}}{2(-\frac{1}{2})}$$

5. [20 points] Find the general solution of the homogeneous equation

$$\underbrace{(x^2 - xy + y^2)}_P dx - \underbrace{x^2}_Q dy = 0.$$

$$P(tx, ty) = (tx)^2 - (tx)(ty) + (ty)^2 = t^2(x^2 - xy + y^2) = t^2 P(x, y)$$

$$Q(tx, ty) = -(tx)^2 = t^2(-x^2) = t^2 Q(x, y)$$

This is a homogeneous equation of degree 2.

Let  $y = xv$ .

$$[x^2 - x(xv) + (xv)^2] dx - x^2(dx v + x dv) = 0$$

$$\cancel{x^2} (1 - v + v^2) dx - \cancel{x^2} (dx v + x dv) = 0$$

$$(v^2 - v + 1) dx - (dx v + x dv) = 0$$

$$(v^2 - 2v + 1) dx - x dv = 0$$

$$(v^2 - 2v + 1) dx = x dv$$

$$\frac{1}{x} dx = \frac{1}{v^2 - 2v + 1} dv$$

$$\int \frac{1}{x} dx = \int \frac{1}{v^2 - 2v + 1} dv$$

$$\int \frac{1}{x} dx = \int \frac{1}{(v-1)^2} dv$$

$$\ln|x| = -\frac{1}{v-1} + c$$

$$\ln|x| = -\frac{1}{\frac{y}{x}-1} + c$$

$$c - \ln|x| = \frac{1}{\frac{y}{x}-1}$$

$$\frac{1}{c - \ln|x|} = \frac{y}{x} - 1$$

$$\frac{1}{c - \ln|x|} + 1 = \frac{y}{x}$$

$$y = \frac{x}{c - \ln|x|} + x$$