1) Solve the initial value problem.

$$x'' + 7x' - 18x = 0, \quad x(0) = 1, x'(0) = 13$$

$s^2 + 7s - 18 = 0$

$(s+9)(s-2) = 0$

$s = -9, 2$

$x = Ae^{-9t} + Be^{2t}$

$x(0) = 1$

$1 = A + B$

$x' = -9Ae^{-9t} + 2Be^{2t}$

$x'(0) = 13$

$13 = -9A + 2B$

$11 = -11A$

$A = -1$

$B = 2$

$x = -e^{-9t} + 2e^{2t}$

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2) A 2 kg mass is attached to a horizontal spring with spring constant 20 N/m. The damping constant is 4 kg/s. Let x(t) denote the position of the mass (in meters) after t seconds. (Here x = 0 m corresponds to the equilibrium position of the spring.) Suppose at time t = 0 the position of the mass is 0 and the velocity is 6 m/s. Find the position of the mass at time t = π/6 s.

$m = 2$

$k = 20$

$b = 4$

$mx'' + bx' + kx = 0$

$2x'' + 4x' + 20x = 0$

$x'' + 2x' + 10x = 0$

$s^2 + 2s + 10 = 0$

$s = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$

$x = e^{-t}(A\cos(3t) + B\sin(3t))$

$x(0) = 0$

$0 = A$

$x = Be^{-t}\sin(3t)$

$x' = -Be^{-t}\sin(3t) + 3Be^{-3t}\cos(3t)$

$x'(0) = 6$

$6 = 3B$

$B = 2$

$x = 2e^{-t}\sin(3t)$

$x(\frac{\pi}{6}) = 2e^{-\frac{\pi}{6}}\sin(3(\frac{\pi}{6})) = 2e^{-\frac{\pi}{6}}$

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3a) Find a basis, β, of eigenvectors for the matrix A.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}, \quad \beta =$$

$$\det\left(\begin{bmatrix} 3-\lambda & 4 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)(-3-\lambda) + 8 = 0$$

$$\lambda^2 - 9 + 8 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = 1, -1$$

$\lambda = 1$ $\quad \ker\left(\begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}\right)$ $\quad \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$= \mathrm{span}\left(\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}\right)$

$\lambda = -1$ $\quad \ker\left(\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}\right)$ $\quad \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $= \mathrm{span}\left(\left\{\begin{bmatrix} \right.\right.$

$$\beta = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

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b) Let γ denote the standard basis. Compute the following matrices. (Hint: The computations should be very short!)

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${}_\gamma[Id]_\beta =$ $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

${}_\beta[Id]_\gamma =$ $\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$

${}_\beta[A]_\beta =$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4) Find the general solution to the differential equation.

$$x'' - 2x' + x = 4e^t$$

try $x_p = ae^t$

$a e^t - 2ae^t + ae^t = 4e^t$

$0 = 4e^t$

exceptional case

try $x_p = ate^t$

$x_p' = ae^t + ate^t$

$x_p'' = 2ae^t + ate^t$

$2ae^t + ate^t - 2ae^t - 2ate^t + ate^t = 4e^t$

$0 = 4e^t$

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exceptional case

try $x_p = at^2e^t$

$x_p' = 2ate^t + at^2e^t$

$x_p'' = 2ae^t + 2ate^t + 2ate^t + at^2e^t$

$= 2ae^t + 4ate^t + at^2e^t$

$2ae^t + 4ate^t + at^2e^t - 4ate^t - 2at^2e^t + at^2e^t = 4e^t$

$2ae^t = 4e^t$

$a = 2$

$x_p = 2t^2e^t$

$r^2 - 2r + 1 = 0$

$r = 1$

$\boxed{x = Ae^t + Bte^t + 2t^2e^t}$