1) Solve the differential equation. If possible, find an explicit solution for y as a function of x. Otherwise, find an implicit solution. (Hint: Suppose there is an integrating factor which is a function of y alone.)

$$xy^2dx + (1+3x^2y)dy = 0$$

$\frac{d}{dy}(\mu x y^2) = \mu' x y^2 + 2\mu x y$

$\frac{d}{dx}(\mu(1+3x^2y)) = 6\mu x y$

$\mu' x y^2 + 2\mu x y = 6\mu x y$

$\mu' x y^2 = 4\mu x y$

5

$\frac{d\mu}{dy} = \frac{4\mu}{y}$

$\frac{d\mu}{4\mu} = \frac{1}{y}dy$

$\frac{1}{4}\ln|\mu| = \ln|y|$

$\mu = y^4$ ✓

$xy^6dx + (y^4 + 3x^2y^5)dy = 0$

$\int xy^6 dx = \frac{1}{2}x^2y^6 + \phi(y)$

$\frac{d}{dy}\left(\frac{1}{2}x^2y^6 + \phi(y)\right) = y^4 + 3x^2y^5$

$3x^2y^5 + \phi'(y) = y^4 + 3x^2y^5$

$\phi'(y) = y^4$

$\phi(y) = \frac{y^5}{5}$

✓

$\boxed{\frac{1}{2}x^2y^6 + \frac{y^5}{5} = C}$

2) A 20°C can of soda is placed inside an empty freezer. When the freezer is opened at t = 0 hours, the incoming warm air raises the temperature inside to 5°C. Once closed, the air temperature inside the freezer (in °C) after t hours is given by $5e^{-2\ln(2)t}$. The constant of proportionality, k, in Newton's Law of Cooling is given by ln(2). Find the temperature of the soda after 1 hour. **Please express your answer as a rational number.**

$$\frac{dT}{dt} = -k(T - A)$$

$A = 5e^{-2\ln(2)t}$

$\frac{dT}{dt} = -\ln(2)\left(T - 5e^{-2\ln(2)t}\right)$

$\frac{dT}{dt} = -\ln(2)T + 5\ln(2)e^{-2\ln(2)t}$

$\frac{dT}{dt} + \ln(2)T = 5\ln(2)e^{-2\ln(2)t}$

$e^{\int \ln(2)dt} = e^{\ln(2)t} = 2^t$

$\frac{d}{dt}(2^tT) = 5\ln(2)(2^{-2t})(2^t)$

$\frac{d}{dt}(2^tT) = 5\ln(2)e^{-\ln(2)t}$

$2^tT = \int 5\ln(2)e^{-\ln(2)t}\,dt$

t is measured in hours

$2^tT = -5e^{-\ln(2)t} + C$

$2^tT = -5\,2^{-t} + C$

$T(60) = -5(2^{-120}) + 25 \cdot 2^{-60}$

$T = -5\,2^{-2t} + C2^{-t}$

$= \frac{-5}{2^{120}} + \frac{25}{2^{60}}$

$T(0) = 20$

$= \boxed{\frac{25 \cdot 2^{60} - 5}{2^{120}}\ °C}$

$20 = -5 + C$

4

$C = 25$

$\boxed{T = (-5)(2^{-2t}) + (25)(2^{-t})}$

3) Solve the initial value problem. If possible, find an explicit solution for y as a function of x. Otherwise, find an implicit solution.

$$(y^2 + 3xy)dx + (xy + x^2)dy = 0, \quad y(1) = 1$$

$y = vx$

$(v^2x^2 + 3vx^2)\,dx + (vx^2 + x^2)(v\,dx + x\,dv) = 0$

$(v^2x^2 + 3vx^2 + v^2x^2 + vx^2)\,dx + (vx^3 + x^3)\,dv = 0$

$(2v^2x^2 + 4vx^2)\,dx + (vx^3 + x^3)\,dv = 0$

$(2x^2(v^2 + 2v))\,dx + (x^3(v+1))\,dv = 0$

$\left(\frac{2}{x}\right)dx + \left(\frac{v+1}{v^2+2v}\right)dv = 0$

5

$\int \frac{2}{x} = 2\ln|x|$ Easier: $u = v^2 + 2v$

$\frac{v+1}{v^2+2v} = \frac{v+2-1}{v^2+2v} = \frac{v+2}{v^2+2v} - \frac{1}{v^2+2v} = \frac{1}{v} - \frac{1}{v^2+2v}$

$\frac{1}{v^2+2v} = \frac{A}{v} + \frac{B}{v+2}$

$A = \frac{1}{2}$

$B = -\frac{1}{2}$

$\frac{1}{v} - \frac{1}{2v} + \frac{1}{2(v+2)}$

$= \frac{1}{2v} + \frac{1}{2(v+2)}$

$\int \frac{1}{2v}\,dv + \int \frac{1}{2(v+2)}\,dv$

$\frac{1}{2}\ln|v| + \frac{1}{2}\ln|v+2|$

$2\ln|x| + \frac{1}{2}\ln|v| + \frac{1}{2}\ln|v+2| = C$

$\frac{1}{2}\ln|(v)(v+2)| = C - 2\ln|x|$

$\ln|(v)(v+2)| = C - 4\ln|x|$

$|(v)(v+2)| = e^c/x^4$

$v^2 + 2v - Ax^{-4} = 0$

$v = \frac{-2 \pm \sqrt{4+4A}}{2}$

$v = -1 \pm \sqrt{1+}$

$\boxed{y = -x \pm x\sqrt{1+}}$

$y(1) = 1$

$1 = -1 \pm 1\sqrt{1 + A \cdot 1}$

$2 = \pm\sqrt{1+A}$

$4 = 1 + A$

$A = 3$

$\boxed{y = -x + x\sqrt{1 + 3x^{-4}}}$

4) A large tank, capable of holding 100 gallons of water, currently holds 1 gallon of saltwater solution, at a concentration of 2 pounds per gallon. At time t = 0, saltwater solution at a concentration of 1 pound per gallon pours into the tank at a rate of 3 gallons per hour. At the same time, a small drain at the bottom of the tank is opened, allowing water to leave the tank at a rate of 2 gallons per hour. What is the mass of the salt in the tank after two hours?

$$\frac{dx}{dt} = \left(1 \frac{\text{pound}}{\text{gallon}}\right)\left(3 \frac{\text{gallon}}{\text{hr}}\right) - \left(\frac{x}{1+t} \frac{\text{pound}}{\text{gallon}}\right)\left(2 \frac{\text{gallon}}{\text{hr}}\right)$$

$$\frac{dx}{dt} = 3 - \frac{2}{1+t}x \checkmark$$

$$\frac{dx}{dt} + \frac{2}{1+t}x = 3 \checkmark$$

$$e^{\int \frac{2}{1+t}dt} = e^{2\ln|1+t|} = (1+t)^2$$

$$(1+t)^2\frac{dx}{dt} + 2(1+t)x = 3(1+t)^2$$

$$\frac{d}{dt}\left((1+t)^2(x)\right) = 3(1+t)^2$$

Easier.

$$3\int(1+t)^2 dt = (1+t)^3 + C$$

$$(1+t)^2(x) = 3\int(1+2t+t^2)\,dt$$

$$(1+t)^2(x) = 3\left(t + t^2 + \frac{t^3}{3}\right) + C$$

5

$$\checkmark\ x = 3\frac{t+t^2+\frac{t^3}{3}}{(1+t)^2} + \frac{C}{(1+t)^2}$$

$$x(0) = 2$$

$$2 = 3(0) + \frac{C}{1}, \quad C = 2$$

$$\boxed{x = 3\frac{t+t^2+\frac{t^3}{3}}{(1+t)^2} + 2/(1+t)^2}$$

$$t=2:\ 3\frac{(2+2^2+\frac{2^3}{3})}{(1+2)^2} + \frac{2}{(1+2)^2}$$

$$\frac{6+12+8+2}{9} = \boxed{\frac{28}{9}} \text{ pounds}$$