Midterm 1

Math 33B-1
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Section (*circle one*): **1A 1B 1C 1D 1E 1F**

Read all of the following information before starting the exam:

- By writing your name on this exam, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- No calculators, cell phones, notes, or other outside materials are permitted on this exam. Do not use your own scratch paper. Scratch paper will be provided upon request. **Do not** turn in any solutions on your scratch paper. It is not to be handed in. All your solutions must be written on the pages of the exam.
- Show all work **clearly** and in order. Justify your answers as much as possible. We reserve the right to take off points if we cannot see how you arrived at your answer (even if your final answer is correct).
- Box or otherwise highlight your final answer. Always try to put your answer in explicit form unless otherwise indicated. You do not need to provide an interval of existence for your solution unless asked to do so.
- If you need additional space for an answer, continue on the back of the page and indicate that the solution continues there.
- This test has ~~4~~ 5 problems and is worth 100 points. Problems 4 and 5 are 2 pages long. It is your responsibility to make sure that you have all of the pages.
- Good luck!

Problem	Score	Possible
1	22	22
2	16	16
3	10	16
4	24	24
5	22	22
Total	94	100

1. (*22 points*) Find the general solution of each differential equation below.

a. (*11 pts*) Give your answer in *explicit* form.

$$y' = \frac{xy^2}{x-1}$$

$\frac{dy}{dx} = \frac{x}{x-1} y^2$

$\int \frac{1}{y^2} dy = \int \frac{x}{x-1} dx$ $\quad u = x-1 \rightarrow x = u+1$, $du = dx$

$-\frac{1}{y} = \int \frac{u+1}{u} du = \int 1 + \frac{1}{u} du$

$= u + \ln u + C$

$-\frac{1}{y} = x - 1 + \ln|x-1| + C$

$\boxed{y = \frac{1}{1 - x - \ln|x-1| + C}}$

OK SHOULD ABSORB 1 TO C

b. (*11 pts*) You may leave your answer in *implicit* form.

$$(\sin y)dx + (x \cos y - 2y)dy = 0$$

$P dx + Q dy = 0$

$\frac{\partial P}{\partial y} = \cos y, \quad \frac{\partial Q}{\partial x} = \cos y \Big]$ exact

$F(x,y) = \int P \, dx = \int \sin y \, dx + \phi(y)$

$= x \sin y + \phi(y)$

$\frac{\partial F}{\partial y} = x \cos y + \phi'(y) \leftarrow -2y$

$\phi(y) = \int -2y \, dy = -y^2$

$F(x,y) = \boxed{x \sin y - y^2 = C}$

2. (*16 points*) Consider the initial value problem:

$$\frac{dx}{dt} = xe^{xt}(x^2 - t), \quad x(0) = -1$$

Without attempting to solve for the solution $x(t)$ to this IVP, show that it must satisfy $x(t) < 0$ for all t for which the solution is defined. State and use a theorem from class.

$x_1(0) = 0$

$\frac{dx}{dt} = 0e^0(0 - t) = 0$, so $x_1(t) = 0$ is a solution

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$f(x,t) = xe^{xt}(x^2 - t)$

f is continuous on the whole xt-plane

$\frac{\partial}{\partial x} xe^{xt} = x^2e^{xt} + e^{xt}$

$\frac{\partial}{\partial x} x^2e^{xt} = x^3e^{xt} + 2xe^{xt}$

$\frac{\partial f}{\partial x} = (x^2e^{xt} + e^{xt})(2x) + (x^3e^{xt} + 2xe^{xt})(x^2 - t)$

$\frac{\partial f}{\partial x}$ is continuous on the whole xt-plane

By theorem 7.16, since $f(x,t)$ and $\frac{\partial f}{\partial x}$ are both continuous over the rectangle $(-\infty, \infty) \times (-\infty, \infty)$, there is a unique solution for any given initial condition.

Since $x(t)$ and $x_1(t) = 0$ are both solutions, they cannot cross and therefore $x(t) < 0$ for all t where the solution is defined.

3. (*16 points*) Consider the differential form $\omega = P(x,y)dx + Q(x,y)dy$. Assume P and Q have continuous partial derivatives.

a. (*6 pts*) Give the definition of what it means for ω to be *exact*.

ω is exact if and only if for a function $F(x,y)$, $\frac{\partial F}{\partial x} = P(x,y)$ and $\frac{\partial F}{\partial y} = Q(x,y)$,

b. (*10 pts*) Using the definition from part (a) and a property from calculus, show that exactness implies that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (as discussed in class).

$\frac{\partial F}{\partial x} = P \rightarrow \partial F = P \partial x$

4

$\frac{\partial F}{\partial y} = Q \rightarrow \partial F = Q \partial y$

so $P \partial x = Q \partial y$

$P = Q \frac{\partial y}{\partial x}$

$\frac{P}{\partial y} = \frac{Q}{\partial x}$

4. (*24 points*) A large tank is partially filled with 100 gal of brine solution, with 0.1 lb/gal of salt. A brine solution with 0.5 lb/gal of salt is poured into the tank at a rate of 4 gal/min. The well-mixed solution is then drained from the tank at the slower rate of 2 gal/min.

a. (*11 pts*) Write out the differential equation that models the amount of salt in the tank. Use x to represent the amount of salt in pounds and t to represent the time in minutes.

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$x' = R_{in} - R_{out}$

$R_{in} = 0.5 \text{ lb/gal} \cdot 4 \text{ gal/min}$

$= 2 \text{ lb/min}$

$R_{out} = \frac{x}{V} \text{ lb/gal} \cdot 2 \text{ gal/min}$ $\leftarrow V = 100 + 2t$

$= \frac{x}{100+2t} \cdot 2 = \frac{x}{50+t} \text{ lb/min}$

$\boxed{x' = 2 - \frac{x}{50+t}}$, $x(0) = 10$

$\frac{dy}{dx} = a(x)y + f(x)$

b. (*11 pts*) Solve the differential equation you found in part (a) to get its general solution.

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$x' = 2 - \frac{x}{50+t}$

$= -\frac{1}{50+t}x + 2$

$x_h = e^{-\int \frac{1}{50+t} dt} = e^{-\ln|50+t|} = \frac{1}{50+t}$

$v' = \frac{f(t)}{x_h} = 2(50+t)$

$\int dv = \int (100 + 2t)\, dt$

$v = 100t + t^2 + C$

$\boxed{x(t) = \frac{100t + t^2 + C}{50+t}}$

c. (*2 pts*) Use the initial condition given in the problem to find the particular solution $x(t)$ that models this system.

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$x(0) = 10$

$x(0) = 10 = \frac{0 + 0 + C}{50 + 0} = \frac{C}{50}$

$C = 500$

$\boxed{x(t) = \frac{100t + t^2 + 500}{50+t}}$

$$a(x)\frac{dy}{dx} = b(x)y + c(x)$$

5. (*22 points*) Consider the first-order differential equation

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

a. (*8 pts*) Determine which of the following characterization(s) this differential equation (or its equivalent differential form) satisfies: *separable, linear, exact, homogeneous of degree n.*

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

not separable, not linear

$$x^2 dy = (x^2 + xy + y^2) dx$$

$$(x^2 + xy + y^2) dx - x^2 dy = 0$$

$$\frac{\partial P}{\partial y} = x + 2y \qquad \frac{\partial Q}{\partial x} = -2x$$

not exact

$$y'(tx, ty) = \frac{t^2x^2 + t^2xy + t^2y^2}{t^2x^2} = \frac{t^2}{t^2} y'(x,y)$$

homogeneous of degree 0

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b. (*14 pts*) Find the general solution to the differential equation. You may leave your answer in implicit form.

$$y' = \frac{x^2 + xy + y^2}{x^2} \quad , \quad (x^2 + xy + y^2)\,dx - x^2\,dy = 0$$

let $y = vx$

$dy = v\,dx + x\,dv$

$(x^2 + \cancel{vx^2} + v^2x^2)\,dx - x^2(\cancel{v\,dx} + x\,dv) = 0$

$(x^2 + v^2x^2)\,dx - x^3\,dv = 0$

$(1 + v^2)\,dx - x\,dv = 0$

$(1+v^2)\,dx = x\,dv$

$\int \frac{1}{x}\,dx = \int \frac{1}{1+v^2}\,dv$

$\ln|x| = \tan^{-1} v + C$

$v = \frac{y}{x}$

$\boxed{\ln|x| = \tan^{-1}\frac{y}{x} + C}$

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