Math 33B Midterm 1 Fall 2011

1. (10 points) Consider the following differential equations:

$$y' + \sin y = t^3 \tag{1}$$

$$\frac{\partial u}{\partial t} = -u^4 - k\frac{\partial u}{\partial x} \tag{2}$$

$$\frac{dx}{dt} + t = t^2 x - 1 \tag{3}$$

$$\frac{dz}{dt} = (z-1)^2 + 3 \tag{4}$$

$$\frac{y'}{y} = \cos x \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 5u + \ln 2 \tag{6}$$

(a) Which of the above are ordinary differential equations?

1, 3, 4, and 5

(b) Which of the above are linear ordinary differential equations?

3 and 5

(c) Of the above linear ODEs, which are homogeneous?

5

(d) Put each of the ordinary differential equations into normal form.

$$y' = -\sin y + t^3$$

PDE

$$\frac{dx}{dt} = t^2 x - t - 1$$

$$\frac{dz}{dt} = (z-1)^2 + 3 \quad \text{or} \quad \frac{dz}{dt} = z^2 - 2z + 4$$

$$y' = \cos x \, y$$

PDE

2. (20 points) Find the general solution to

$$(x^2 + 2y) + (2x - 3y)\frac{dy}{dx} = 0.$$

Express the solution explicitly for $y(x)$ if possible, otherwise an implicit solution is fine.

This is a nonlinear differential equation and is not separable. We can use the exact differential approach to solve this problem.

$$(x^2 + 2y)dx + (2x - 3y)dy = 0$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2,$$

this is an exact differential, therefore a function F exists and such that $dF = Pdx + Qdy = 0$ and $F(x, y) = C$ is a solution to the differential equation.

We solve for F:

$$\frac{\partial F}{\partial x} = P = x^2 + y$$

$$F = \int (x^2 + y^2)dx + \phi(y) = \frac{x^3}{3} + 2xy + \phi(y)$$

We then take the derivative of F w.r.t. y to find $\phi(y)$:

$$\frac{\partial F}{\partial y} = 2x + \phi'(y) = Q = 2x - 3y$$

$$\phi'(y) = -3y$$

$$\phi(y) = -3y^2/2$$

We now have a complete expression for $F(x, y)$, so $F(x, y) = C$ is an implicit solution:

$$F(x, y) = x^3/3 + 2xy - 3y^2/2 = C,$$

or

$$2x^3 + 12xy - 9y^2 = C.$$

3. (20 points) Use the *variation of parameters* method to find the general solution to:

$$y' + y = t.$$

(Note: if you cannot evaluate an integral, then leave the integral in your solution.)

The first step in variation of parameters is to find the solution to the homogeneous equation

$$y' + y = 0.$$

This is separable and can be solved by integration to obtain

$$y_h(t) = e^{-t}.$$

We then substitute the solution, $y(t) = v(t)y_h(t)$ into the governing DE and solve for $v(t)$.

$$vy_h = ve^{-t}$$

$$v'e^{-t} - ve^{-t} = -ve^{-t} + t$$

$$v'e^{-t} = t$$

$$v' = te^t$$

$$v = \int te^t dt + C$$

$$v = e^t(t-1) + C$$

Now that $v(t)$ is known, we have a complete expression for the solution,

$$y = \left(e^t (t-1) + C\right) e^{-t}$$

$$y = t - 1 + Ce^{-t}$$

4. (15 points) Does a solution exist for the following initial value problem? If yes, is it unique? Justify your answer.

$$t\frac{dy}{dt} = y - \frac{t^3}{3}, \qquad y(3) = 0, \qquad t > 0$$

$$y' = f = \frac{y}{t} - \frac{t^2}{3}$$

f is discontinuous at t=0 only. However, our domain is $(0, \infty)$.

Since f is continuous and defined for $t > 0$, we can choose any rectangle, $R : t > 0$ that contains the point $(t_o, y_o) = (3, 0)$, in the ty-plane to apply the existence theorem, therefore a solution exists.

4. *(continued blank space)*

We calculate the derivative of f w.r.t. y,

$$\frac{\partial f}{\partial y} = \frac{1}{t}.$$

It is clear that f and $\partial f / \partial y$ are continuous for all $t > 0$, so we can choose any rectangle, $R : t > 0$, that contains the initial point $(t_o, y_o) = (3, 0)$ in the ty-plane to apply the uniqueness theorem, therefore the solution is unique.

5. (20 points) The Shell oil company wants to add nitrogen to regular gasoline. They claim that "nitrogen enriched" gasoline is better because it cleans your engine. Of course, any scientist who has studied combustion knows that nitrogen is an inert substance, and therefore this the equivalent of "watering-down" the gasoline to increase profits.

   You have been put in charge of the gasoline processing plant. A 1,600 gallon tank initially contains 1000 gallons of pure gasoline. At time $t = 0$, an incoming source of nitrogen-enriched gasoline enters the tank at a rate of 10 gal/min and contains 4lbs/gal of nitrogen. At the same time, an outflow of "nitrogen-enriched gasoline" begins to exit the tank at a rate of 8 gal/min. Assume perfect mixing.

   (a) Write a differential equation that describes the nitrogen content, $x(t)$ [lbs], in the tank.

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 10\frac{\text{gal}}{\text{min}} \times 4\frac{\text{lbs}}{\text{gal}} = 40\frac{\text{lbs}}{\text{min}}$$

$$\text{rate out} = 8\frac{\text{gal}}{\text{min}} \times \frac{x(t)}{v(t)}\frac{\text{lbs}}{\text{gal}}$$

The volume changes with time according to

$$v(t) = 1000 + 2t,$$

therefore the governing differential equation is:

$$\frac{dx}{dt} = 40 - \frac{8x}{1000 + 2t},$$

$$\frac{dx}{dt} = 40 - \frac{4x}{500 + t}.$$

5. *(continued)*
(b) Find the particular solution to the differential equation from part (a).

This is a first-order, linear, inhomogeneous equation so we can use an integrating factor approach or variation of parameters. We show the integrating factor approach in this solution.

We find an integrating factor, $u(t)$, such that

$$(ux)' = uf$$

The integrating factor is given by,

$$u(t) = e^{-\int a(t)dt}$$

$$u = e^{-\int -4/(500+t)dt} = e^{4\ln(500+t)}$$

$$u = (500+t)^4$$

We then multiply both sides of the DE by the integrating factor and solve for $x(t)$:

$$ux = \int uf dt + C$$

$$(500+t)^4 x = 40\frac{(500+t)^5}{5} + C$$

$$x(t) = 8(500+t) + \frac{C}{(500+t)^4}$$

Since the tank initially contains pure gasoline, the initial condition $x(0) = 0$ allows us to solve for the value of the constant, C,

$$x(0) = 0 = 4000 + C(500)^{-4}, \qquad C = -4000 \cdot 500^4,$$

and the particular solution is given by,

$$x(t) = 4000 + 8t - \frac{4000\left(500^4\right)}{(500+t)^4}$$

(c) What is the nitrogen content in the tank at the time in which the tank becomes full? (You may leave the answer as an expression for x without evaluating it.)

The volume changes with time according to

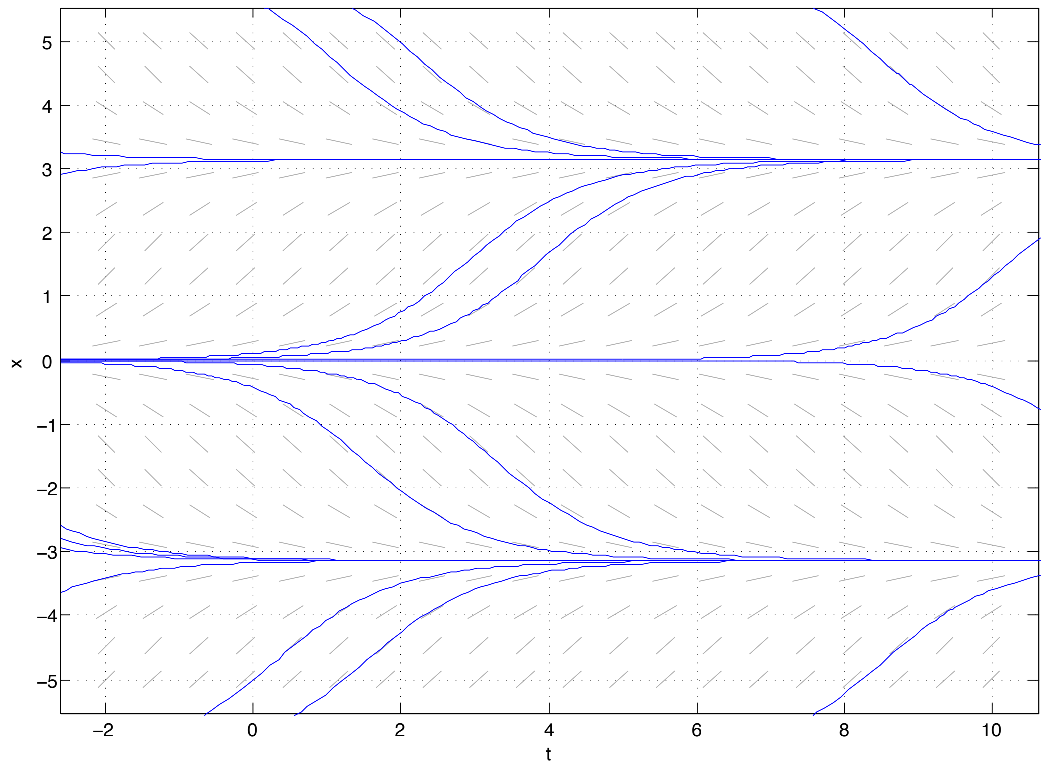
$$v(t) = 1000 + 2t,$$

$$1600 = 1000 + 2t_{\text{full}},$$

$$t_{\text{full}} = 300 \text{ min}$$

$$x(300) = 4000 + 8(300) - 4000(500^4)/(800^4) \approx 5800 \text{ lbs}$$

6. (15 points) Consider the following direction field plot:



(a) Sketch several solution curves on the direction field plot above.

(b) Is the differential equation autonomous (provide justification for your answer)? If so, draw a phase line diagram (you may approximate the locations of any equilibrium points).

Yes, the differential equation is autonomous because the direction field does not change with time. The phase line is shown below with asymptotically-stable equilibrium points at $x = -\pi$ and $x = \pi$, and an unstable equilibrium point at $x = 0$.