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1. (16 points)
   (a) Define the Wronskian W_{y_1,y_2} of two functions y_1 and y_2
   (b) Define what it means for two functions y_1,y_2 to be linearly dependent
   (c) Show that $W_{y_1,y_2} = 0$ if y_1, y_2 are linearly dependent

a) The Wronskian $W_{y_1 y_2}$ is the $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ ~~it~~ indicates whether the two functions y_1 and y_2 are linearly dependent, which occurs if $W_{y_1 y_2} = 0$ or if ~~the~~ the two functions are linearly independent, which occurs if $W_{y_1,y_2} \neq 0$. 4

b) If the two functions y_1, y_2 are linearly dependent, there there is some constant A and some constant B ~~such~~ that $y_1 = Ay_2$ and $By_1 = y_2$. In other words if the two functions are linearly dependent, then one function is a scalar multiple of the other function. 2

what if $y_1 = 0$ or $y_2 = 0$?

$c_1 y_1 + c_2 y_2 = 0$
c_1, c_2 are not both 0

~~or~~ $W_{y_1 y_2} = 0$

c) Suppose $y_1 = Ay_2$, meaning that y_1 is a scalar multiple of y_2 and by the definition, y_1 and y_2 are linearly dependent. 6

$$W_{y_1 y_2} = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \to \det\begin{pmatrix} Ay_2 & y_2 \\ Ay_2' & y_2' \end{pmatrix} = Ay_2y_2' - Ay_2y_2' = 0$$

Suppose $By_1 = y_2$ ... why can ??

$$W_{y_1 y_2} = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \to \det\begin{pmatrix} y_1 & By_1 \\ y_1' & By_1' \end{pmatrix} = By_1y_1' - By_1y_1' = 0$$

Therefore $W_{y_1 y_2} = 0$ if y_1 and y_2 are linearly dependent.

$c_1 y_1 + c_2 y_2 = 0$
$c_1 \neq 0$
$c_2 y_2 = -c_1 y_1$
$y_1 = -\frac{c_2}{c_1} y_2$
then plug into Wronskian

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2. (20 points) Find the general solution of the following equation.

$$t^2y'' - 2y = 0 \quad \text{for } t > 0$$

(a) Find the general solution to the equation.

(b) Find a particular solution to the equation satisfying the initial value conditions $y(2) = 3$, $y'(-1) = -3$.

(c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

a) $t^2y'' - 2y = 0$ for $t > 0$

Euler Equation

Let $y = t^r$

then the characteristic polynomial is $r^2 - r - 2 = 0$

$(r-2)(r+1)$

Therefore, $y_1 = t^2$ $y_2 = t^{-1}$

$y_1 = t^2$ $y_2 = t^{-1}$
$y_1' = 2t$ $y_2' = -t^{-2}$
$y_1'' = 2$ $y_2'' = 2t^{-3}$
$2t^2 - 2t^2 = 0$ ✓ $2t^{-1} - 2t^{-1} = 0$ ✓

$W_{y_1, y_2} = \det\begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix}$

$-1 - 2 \neq 0$

The general solution is $y(t) = c_1t^2 + c_2t^{-1}$ ✓

The general solution is $y(t) = c_1t^2 + c_2t^{-1}$ where c_1 + c_2 are constants

b) $y(2) = 3$ $y'(-1) = -3$

$y(2) = c_1(2)^2 + c_2(\frac{1}{2}) = 3$

$4c_1 + \frac{1}{2}c_2 = 3$

$8c_1 + c_2 = 6$
$-2c_1 - c_2 = -3$
$6c_1 = 3$
$c_1 = \frac{1}{2}$

$y'(t) = 2c_1t - c_2t^{-2}$

$y'(-1) = 2c_1(-1) - c_2(-1)^{-2} = -3$

$-2c_1 - c_2 = -3$

$-2(\frac{1}{2}) - c_2 = -3$

$-1 - c_2 = -3$

$c_2 = 2$

$y(t) = \frac{1}{2}t^2 + 2t^{-1}$ $y(t) = \frac{1}{2}t^2 + 2t^{-1}$ ✓

c) It is not possible to apply the Uniqueness and Existence Theorem. The Uniqueness + Existence Theorem requires that the same t_0 is used, in other terms ~~so that~~ $y(t_0) = y_0$ and $y'(t_0) = y_1$. ~~A solution can be found but it is not certain if the solution is unique~~. If different values are plugged in, it is uncertain if a solution exists or if it is unique.

3. (18 points) Consider the following differential equation

$$y'' - 2y' + y = \frac{e^t}{t} \quad \text{for } t > 0$$

(a) Find the general solution to the associated homogeneous equation.

(b) Find the general solution to the given inhomogeneous equation.

a) $y_H: \; y'' - 2y' + y = 0$

characteristic polynomial: $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 \rightarrow y_1 = e^t \quad y_2 = te^t$

$W_{y_1 y_2} = \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}$

$e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}}$

$e^{2t} \neq 0$

$y_H(t) = c_1 e^t + c_2 te^t$

$y_H(t) = c_1 e^t + c_2 te^t$ $\quad c_1, c_2$ are constants

b) $y_p = v_1 y_1 + v_2 y_2$

$y_1 = e^t, \; y_2 = te^t$

$v_1 = -\int \frac{\cancel{e^t}}{\cancel{t} \cdot \cancel{e^{2t}}} (\cancel{t}\cancel{e^t})\, dt$

$-\int dt$

$-t + c_3$ $\quad c_3 = 0$ since we want a particular solution

$v_2 = \int \frac{\cancel{e^t}}{t\, \cancel{e^{2t}}} (\cancel{e^t})\, dt$

$\int \frac{1}{t}\, dt$

$\ln(t) + c_4$ $\quad c_4 = 0$ since we want a particular solution

te^t

$e^t + te^t$

$2e^t + te^t$

$\cancel{2e^t} + \cancel{te^t} - \cancel{2e^t} - \cancel{2te^t}$

$+ \cancel{te^t}$

$te^t \ln t$

$(e^t + te^t)\ln t + \frac{1}{t}(te^t)$

$e^t \ln t + \frac{1}{t}e^t + \; + e^t$

$(e^t + te^t)\ln t + e^t$

$2e^t \cancel{\ln t} + \frac{1}{t}e^t + \cancel{2e^t} + \cancel{te^t}\ln t$

$-\cancel{2e^t}\ln t - 2te^t\cancel{\ln t} - \cancel{2e^t}$

$+ te^t \cancel{\ln t}$

$y_p = -te^t + te^t(\ln t)$

$y(t) = c_1 e^t + \underbrace{c_2 te^t - te^t}_{\text{can be combined through } c_2} + te^t(\ln t)$

$y(t) = c_1 e^t + c_2 te^t + te^t \ln(t)$ $\quad c_1, c_2$ are ~~arbitrary~~ constants

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4. (16 points) Consider the following differential equation

$$y'' + 2y = e^t \sin(t)$$

(a) Find the general solution to the associated homogeneous equation.
(b) Find the general solution to the given inhomogeneous equation.

a) $y_H = y'' + 2y = 0$

characteristic polynomial : $\lambda^2 + 2 = 0$

$\lambda = \pm\sqrt{-2}$

$\lambda = \pm\sqrt{2}\, i$

$y_1 = \cos(\sqrt{2}t)$

$y_2 = \sin(\sqrt{2}t)$

$W_{y_1,y_2} = \det\begin{pmatrix} \cos\sqrt{2}t & \sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \sqrt{2}\cos\sqrt{2}t \end{pmatrix}$

$\sqrt{2}\cos^2\sqrt{2}t + \sqrt{2}\sin^2\sqrt{2}t$

$W_{y_1,y_2} = \sqrt{2}$

$y_H(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t)$

$y_H(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t)$, c_1, c_2 ~~and~~ are constants

b) $y'' + 2y = e^t\sin(t)$

$y_p = e^t e^{it}$ with e^{it} imag. sol

e^{t+it}

$y_p = v_0 e^t e^{it}$ imag portion

$y_p' = v_0(e^t e^{it} + ie^t e^{it})$

$y_p'' = v_0(e^t e^{it} + ie^t e^{it} + ie^t e^{it} + i^2 e^t e^{it})$

$v_0(\cancel{e^t e^{it}} + 2ie^t e^{it} - \cancel{e^t e^{it}})$

$v_0(2ie^t e^{it} + 2e^t e^{it}) = e^t e^{it}$

$v_0(2i+2) = 1$

$v_0 = \frac{1}{2i+2}\frac{(-2i+2)}{(-2i+2)} = \frac{-2i+2}{4+4} = \frac{-2i+2}{8} = \frac{-i+1}{4}$

$y_p = \left(\frac{-i+1}{4}\right)\left[e^t\right](i\sin t + \cos t)$

$y_p = e^t\left[\frac{1}{4}\sin t + \frac{i}{4}\sin t + \frac{-i}{4}\cos t + \frac{1}{4}\cos t\right]$

$y_p = \frac{1}{4}(e^t\sin t - e^t\cos t)$

$y(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + \frac{1}{4}e^t\sin t - \frac{1}{4}e^t\cos t$

$y(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + \frac{1}{4}e^t\sin t - \frac{1}{4}e^t\cos t$