14

1. (16 points)
   (a) Define the Wronskian W_{y_1,y_2} of two functions y_1 and y_2
   (b) Define what it means for two functions y_1,y_2 to be linearly dependent
   (c) Show that $W_{y_1,y_2} = 0$ if y_1, y_2 are linearly dependent

a) $W_{y_1 y_2} = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

b) y_1, y_2 linearly ~~in~~dependent $\Longleftrightarrow$ ~~the only~~ there exists a solution to $c_1 y_1 + c_2 y_2 = 0$ is non-trivial (i.e. ~~[illegible]~~ $c_1 \neq 0$ or $c_2 \neq 0$ ~~[illegible]~~)

$\Longleftrightarrow \exists\, c \in \mathbb{R}\,|_{c \neq 0}$ such that $y_1 = Cy_2$

c) let $y_1 = Cy_2 \Rightarrow y_1' = Cy_2'$

$W_{y_1,y_2} = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' \ast - y_1' y_2$

$\ast$ substitute

$Cy_2(y_2') - Cy_2'(y_2) = 0$

if $y_2 = Cy_1^2$

20

2. (20 points) Find the general solution of the following equation.

$$t^2y'' - 2y = 0 \quad \text{for } t > 0$$

(a) Find the general solution to the equation.

(b) Find a particular solution to the equation satisfying the initial value conditions $y(2) = 3$, $y'(-1) = -3$.

(c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

a) $t^2y'' + 0ty' - 2y = 0$ * Guess $y = t^r \Rightarrow y' = rt^{r-1} \Rightarrow y'' = (r^2-r)t^{r-2}$

* substitute: $t^r(r^2+(p-1)r+q) = 0 \Rightarrow r^2+(p-1)r+q = 0$

$r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r = 2, -1$

$\therefore \; y = c_1t^2 + c_2t^{-1}$

b) $y(2) = 3 = c_1(2)^2 + c_2(2)^{-1} = 4c_1 + \frac{1}{2}c_2$

$y' = 2c_1t - c_2t^{-2}$

$y'(-1) = -3 = 2c_1(-1) - c_2(-1)^{-2} = -2c_1 - c_2$

$2\times(4c_1 + \frac{1}{2}c_2 = 3)$

$+ \quad -2c_1 - c_2 = -3$

$6c_1 = 3$

$\Rightarrow c_1 = \frac{1}{2}$

$-3 = -2(\frac{1}{2}) - c_2$

$-2 = -c_2$

$\Rightarrow c_2 = 2$

$y_p = \frac{1}{2}t^2 + 2t^{-1}$

c) No; the hypothesis condition that the IVCs for y and y' must be given at the same t_0 is not satisfied (i.e. $2 \neq -1$ in $y(2)=3, y'(-1)=-3$)

3. (18 points) Consider the following differential equation

$$y'' - 2y' + y = \frac{e^t}{t} \quad \text{for } t > 0$$

(a) Find the general solution to the associated homogeneous equation.

(b) Find the general solution to the given inhomogeneous equation.

a) $y'' - 2y' + y = 0$ * guess $y = e^{\lambda t}$

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$, (mult 2)

$\therefore y_h = c_1 e^t + c_2 t e^t$

b) use variation of parameters to find y_p $\left(\begin{array}{l} y_1 = e^t, y_2 = te^t \\ f(t) = \frac{e^t}{t} \end{array} \right)$

$y_p = v_1 y_1 + v_2 y_2$

$W = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^t(e^t(t+1)) - te^{2t}$
$= e^{2t}$

$v_1 = -\int \frac{f}{W} y_2 \, dt = -\int \frac{e^t}{t(e^{2t})}(te^t) \, dt = -\int dt = -t$

$v_2 = \int \frac{f}{W} y_1 \, dt = \int \frac{e^t}{t(e^{2t})}(e^t) \, dt = \int \frac{1}{t} dt = \ln|t|$

* substitute: $y_p = (-t)(e^t) + (\ln|t|)(te^t)$?

$y = y_h + y_p = c_1 e^t + c_2 te^t - te^t + \ln|t| e^t$

* absorb -1 into c_2

$y = c_1 e^t + c_2 te^t + \ln|t| e^t \cdot t$.

-2.

13

4. (16 points) Consider the following differential equation

$$y'' + 2y = e^t \sin(t)$$

6 (a) Find the general solution to the associated homogeneous equation.

7 (b) Find the general solution to the given inhomogeneous equation.

a) $y'' + 2y = 0$; * guess $y = e^{\lambda t}$

$\Rightarrow \lambda^2 + 2 = 0 \Rightarrow \lambda = \pm i\sqrt{2}$) Euler's formula

$y_h = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t)$

b) * Guess $y_p = ae^t \cos t + be^t \sin t$

$\Rightarrow y_p' = ae^t(-\sin t) + (ae^t)\cos t + be^t(\cos t) + (be^t)\sin t$
$= e^t((a+b)\cos t + (-a+b)\sin t)$

$\Rightarrow y_p'' = e^t(-(a+b)\sin t + (-a+b)\cos t) + e^t(a\cos t + b\sin t)$
$= e^t(-a\sin t + b\cos t)$

*substitute

$\cancel{e^t}(-a\sin t + b\cos t) + 2\cancel{e^t}(a\cos t + b\sin t) = \cancel{e^t}\sin t$

$(a+b)\cos t + (-a+b)\sin t = \sin t$

$\Rightarrow a+b = 0, \quad -a+b = 1 \Rightarrow a = -\frac{1}{2}, \; b = \frac{1}{2}$

$\therefore y_p = -\frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t$

$y = y_h + y_p = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + \frac{1}{2}e^t(\sin t - \cos t)$