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Math 33B, Lecture 2
Spring 2018
05/21/18
Time Limit: 50 Minutes

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Day \ T.A.	Blaine	Frank	Siting
Tuesday	2A	2C	2E
Thursday	~~2B~~	2D	2F

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and **cross the box** corresponding to your discussion section.
2. Use a **PEN** to record your final answers.
3. If you need **more space**, use the back of this page and pages 6,8.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	18	18
2	18	18
3	20	17
4	14	9
Total:	70	62

1. (18 points) Consider the following differential equation

$$y'' - 2y' + 2y = e^t \sin(t)$$

(a) Find the general solution to the associated homogeneous equation.

(b) Find the general solution to the given inhomogeneous equation.

(a) $z'' - 2z' + 2z = e^{(i+1)t}$ $\qquad e^t \sin(t) = \mathrm{im}\left(e^{(1+i)t}\right)$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot 1\cdot 2}}{2} = 1 \pm \frac{\sqrt{-4}}{2} = 1 \pm i$

homogeneous soln: $y_1(t) = e^t\cos(t)$, $y_2(t) = e^t \sin(t)$

$\boxed{\text{(a) } y_H(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t), \quad c_1, c_2 \in \mathbb{R} \text{ arbt.}}$

(b) undetermined coefficients

$\alpha = i+1$ is a simple root $\rightarrow$ guess $z_p = u_0 \cdot te^{(i+1)t}$

$z_p' = u_0 \cdot \left((i+1)te^{(i+1)t} + e^{(i+1)t}\right) = u_0 e^{(i+1)t}(it + t + 1)$

$z_p'' = u_0 \cdot \left((i+1)\left((i+1)te^{(i+1)t} + e^{(i+1)t}\right) + (i+1)e^{(i+1)t}\right)$

$= u_0 \cdot \left((i+1)^2 te^{(i+1)t} + (i+1)e^{(i+1)t} + (i+1)e^{(i+1)t}\right)$

$= u_0 \cdot \left((-1+2i+1)te^{(i+1)t} + 2(i+1)e^{(i+1)t}\right) = u_0 e^{(i+1)t}(2it + 2i + 2)$

$z_p'' - 2z_p' + 2z_p = u_0 e^{(i+1)t}\left[2it + 2i + 2 - 2(it + t + 1) + 2(t)\right] = e^{(i+1)t}$

$u_0\left[2it + 2i + \cancel{2} - 2it \cancel{- 2t} \cancel{- 2} \cancel{+ 2t}\right] = 1$

$u_0\left[2i\right] = 1 \rightarrow u_0 = \frac{1(-i)}{2i(-i)} = \frac{-i}{2}$

$z_p = \frac{-i}{2} te^{(i+1)t}$ $\qquad y_p = \mathrm{im}(z_p) = \mathrm{im}\left(\frac{-i}{2} t e^t \cos(t) + \frac{-i}{2} t \cdot ie^t \sin(t)\right)$

$y_p = \frac{-1}{2} te^t \cos(t)$

$\boxed{\text{(b) } y(t) = y_p + y_H = \frac{-1}{2} te^t \cos(t) + c_1 e^t \cos(t) + c_2 e^t \sin(t), \quad c_1, c_2 \in \mathbb{R} \text{ arbt.}}$

2. (18 points) Consider the following differential equation

euler

$$t^2y'' - 3ty' + 4y = 0 \quad \text{solve for } t > 0$$

(a) Find two linearly independent solutions of the equation. Show that the solutions are linearly independent.

(b) What is the general solution of the equation. Justify your answer.

$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0 \qquad p = -3,\ q = 4$

t^r, r = roots of $x^2 + (p-1)x + q = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2, 2$

$y_1(t) = t^2,\ y_2(t) = t^2 \ln(t)$

$W_{y_1,y_2} = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} t^2 & t^2\ln(t) \\ 2t & t^2\cdot\frac{1}{t} + 2t\ln(t) \end{pmatrix}$

$= \det\begin{pmatrix} t^2 & t^2\ln(t) \\ 2t & t + 2t\ln(t) \end{pmatrix} = t^3 + 2t^3\ln(t) - 2t^3\ln(t) = t^3 \neq 0$

$\therefore y_1, y_2$ are lin indp.

~~(a) $y_1(t) = t^{-1}\cos(\sqrt{3}\ln(t))$, $y_2 = t^{-1}\sin(\sqrt{3}\ln(t))$ are 2 lin indp solns~~

$\boxed{\text{(a) } y_1(t) = t^2,\ y_2(t) = t^2\ln(t) \text{ are two lin indp solns}}$

(b) general sol'n $y(t) = c_1 y_1 + c_2 y_2$, $c_1, c_2 \in \mathbb{R}$ arbt

because y_1 & y_2 are lin indp, and L is linear

$\boxed{y(t) = c_1 t^2 + c_2 t^2 \ln(t),\ c_1, c_2 \in \mathbb{R} \text{ arbt}}$

3. (20 points) Consider the equation

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$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}}, \quad \text{for } x > 0$$

We are told that the functions $y_1 = x^{-\frac{1}{2}}\sin(x)$ and $y_2 = x^{-\frac{1}{2}}\cos(x)$ are linearly independent solutions of the associated homogeneous equation.

12/12 (a) Find the general solution of the given equation.

0/3 (b) Find a particular solution to the equation satisfying the initial value conditions $y(\pi/2) = 0$, $y'(\pi) = 0$.

5/5 (c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

(a) $y'' + (\frac{1}{x})y' + (1 - \frac{1}{4x^2})y = x^{-1/2}$

$y_p = v_1 y_1 + v_2 y_2 \qquad W_{y_1,y_2} = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} x^{-1/2}\sin(x) & x^{-1/2}\cos(x) \\ -\frac{1}{2}x^{-3/2}\sin(x) + x^{-1/2}\cos(x) & -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin(x) \end{pmatrix}$

$= -\frac{1}{2}x^{-2}\sin(x)\cos(x) - x^{-1}\sin^2(x) + \frac{1}{2}x^{-2}\cos(x)\sin(x) - x^{-1}\cos^2(x) = -x^{-1}(\sin^2(x) + \cos^2(x)) =$

$= -x^{-1}$

$v_1 = -\int \frac{f}{w} y_2 \, dx = -\int \frac{x^{-1/2}}{-x^{-1}} \cdot x^{-1/2}\cos(x)\, dx = \int \cos(x)\, dx = \sin(x)$

$v_2 = \int \frac{f}{w} y_1 \, dx = \int \frac{x^{-1/2}}{-x^{-1}} \cdot x^{-1/2}\sin(x)\, dx = -\int \sin(x)\, dx = \cos(x)$

$y_p = \sin(x) \cdot x^{-1/2}\sin(x) + \cos(x) \cdot x^{-1/2}\cos(x) = x^{-1/2}(\sin^2(x) + \cos^2(x)) = x^{-1/2}$

(a) $y(t) = y_p + y_H = x^{-1/2} + c_1 x^{-1/2}\sin(x) + c_2 x^{-1/2}\cos(x)$, $c_1, c_2 \in \mathbb{R}$ arbt.

(b) $y(\frac{\pi}{2}) = \sqrt{\frac{2}{\pi}}(1 + c_1 \cdot 1 + c_2 \cdot 0) = 0 \qquad y'(x) = x^{-1/2}(c_1\cos(x) - c_2\sin(x)) + \frac{-1}{2}x^{-3/2}(1 + c_1\sin(x) + c_2\cos(x))$

$c_1 = -\sqrt{\frac{2}{\pi}}$ ✗ $\qquad y'(\pi) = \sqrt{\frac{1}{\pi}}(-1 c_1 - 0 c_2) + \frac{-1}{2}\sqrt{\frac{1}{\pi^3}}(1 + 0 c_1 + -c_2)$

$= -\sqrt{\frac{1}{\pi}}c_1 = 0 \rightarrow c_1 = 0$ ✗

contradicts so

(b) ~~not enough information~~ no solution satisfies these initial conditions ✗

(c) No, the theorem only applies when both initial conditions are at the same t-value t_0. Here we have $\frac{\pi}{2}$ and π which are different.

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4. (14 points)

(a) Write the **definition** of what it means for three functions $y_1(t), y_2(t), y_3(t)$ to be linearly independent.

(b) Let y_1, y_2 be solutions to the innhomogeneous linear ODE $y'' - y = f(x)$. We are told that $y_1(0) = 0, y_1'(0) = 2$, $y_2(0) = 2$, $y_2'(0) = 2$, and $y_1(1) = 1$. Find $y_2(1)$. Justify your answer.

(a) The only combination $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t)$, $c_1, c_2, c_3 \in \mathbb{R}$ satisfying $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) \equiv 0$ is when $c_1 = c_2 = c_3 = 0$. ✓ 9

(b) ~~$y_1''(0) - y_1(0) = f(0)$~~

~~$y_1''(0) =$~~ $f(0)$

$W_{y_1, y_2}' = -pW$

$y_1(0) y_2'(0) - y_2(0) y_1'(0) = 0 \cdot 2 - 2 \cdot 2 = -4 = W(0)$

$L[y_1] = L[y_2] = f(x)$

$L[y_1 - y_2] = f(x) - f(x) = 0$

$\boxed{\text{(b)} \quad y_2(1) = 3}$ 0

$y_1(0) + f(0) = y_1''(0) = 0 + f(0)$

$y_2(0) + f(0) = y_2''(0) = 0 + f(0)$

so $y_2(0) - y_1(0) = y_2(1) - y_1(1) = 2$

$y_1(1) + (y_2(1) - y_1(1)) = y_2(1) = 1 + 2 = 3$