18

Math 33B, Lecture 2
Spring 2018
05/21/18
Time Limit: 50 Minutes

Name (Print): Hannah Park
SID Number: 204 916 327

Signature: H Park

Day \ T.A.	Blaine	Frank	Siting
Tuesday	2A	2C	2E
Thursday	2B	2D	2F

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and **cross the box** corresponding to your discussion section.
2. Use a **PEN** to record your final answers.
3. If you need **more space**, use the back of this page and pages 6,8.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	18	18
2	18	18
3	20	20
4	14	11
Total:	70	67

ath 33A HW 4

Hannah Park

05/21/18

$(1+i)(1+i) = 1+2i-1 = 2i$

1. (18 points) Consider the following differential equation

$$y'' - 2y' + 2y = e^t \sin(t)$$

$z''-2z'+2z = e^{(1+i)t}$

(a) Find the general solution to the associated homogeneous equation.
(b) Find the general solution to the given inhomogeneous equation.

a) Characteristic polynomial $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda = \frac{2 \pm \sqrt{4-4(1)(2)}}{2}$

$= \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$

$\lambda = 1 \pm i$ (2 complex roots)

$y_1 = e^t \cos(t)$, $y_2 = e^t \sin(t)$ — y_1 and y_2 are linearly independent solutions.

$\boxed{y(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)}$ ✓ c_1 and c_2 arbitrary $\in \mathbb{R}$

b) Find $y_p \rightarrow$ Guess with $z''-2z'+2z = e^{(1+i)t}$ and $\alpha = (1+i)$ is a root

do complex method

so guess $z_p(t) = ate^{(1+i)t}$

$z_p' = ae^{(1+i)t} + at(1+i)e^{(1+i)t}$

$z_p'' = a(1+i)e^{(1+i)t} + a(1+i)e^{(1+i)t} + at(1+i)^2 e^{(1+i)t}$

$z_p'' - 2z_p' + 2z_p = e^{(1+i)t}\left(a(1+i) + a(1+i) + at(1+i)^2 - 2a - 2at(1+i) + 2at\right)$

$= e^{(1+i)t}\left(2a(1+i) + at(1+i)^2 - 2a - 2at - 2ait + 2at\right)$

$= e^{(1+i)t}\left(\cancel{2a} + 2ai + \cancel{at(2i)} \cancel{-2a} \cancel{-2at} \cancel{-2ait} \cancel{+2at}\right) = e^{(1+i)t}(2ai)$

Set this equal to $f(t) = e^{(1+i)t}$ so $\cancel{e^{(1+i)t}} = \cancel{e^{(1+i)t}}(2ai) \rightarrow 1 = 2ai$

$a = \frac{1}{2i}$

$a = \frac{1}{2i}\cdot\frac{2i}{2i} = \frac{2i}{(2i)^2} = \frac{2i}{-4} = \frac{-i}{2}$

So $z_p(t) = \frac{-i}{2} \cdot t \cdot e^{(1+i)t} = \left(\frac{-it}{2}\right)\cdot\left(e^t \cos t + e^t i \sin t\right)$

$y_p(t) = Im(z_p(t)) = -\frac{t}{2} \cdot e^t \cdot \cos t$

$y(t) = y_H + y_p$

$\boxed{y(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t) + -\frac{t}{2} e^t \cdot \cos t}$, c_1 and c_2 arbitrary constants $\in \mathbb{R}$

2. (18 points) Consider the following differential equation

$$t^2y'' - 3ty' + 4y = 0 \quad \text{solve for } t > 0$$

(a) Find two linearly independent solutions of the equation. Show that the solutions are linearly independent.

(b) What is the general solution of the equation. Justify your answer.

a) Euler → $p = -3,\ q = 4$

$r^2 + (p-1)r + q = r^2 - 4r + 4 = (r-2)(r-2) \rightarrow r = 2$ double root

$\boxed{y_1 = t^2 \quad y_2 = t^2 \cdot \ln(t)}$

y_1, y_2 linearly independent if $W_{y_1,y_2}(t) \neq 0$ so $W_{y_1,y_2}(t) = \det\begin{pmatrix} t^2 & t^2\ln t \\ 2t & 2t\ln t + \frac{t^2}{t} \end{pmatrix}$

$= t^2 \cdot (2t \cdot \ln t + t) - t^2 \ln t (2t) = 2t^3 \ln t + t^3 - t^2 \ln t (2t)$

$= 2t^3 \ln t + t^3 - 2t^3 \ln t = t^3 \neq 0$ so y_1, y_2 linearly independent.

b) General solution

$y(t) = C_1 y_1 + C_2 y_2$, where y_1 and y_2 are linearly independent solutions to the 2nd order homogeneous ~~linear~~ differential equations. Since $W_{y_1,y_2}(t) \neq 0$ y_1 and y_2 are linearly independent so

$\boxed{y(t) = C_1 t^2 + C_2 t^2 \cdot \ln(t) \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants} \in \mathbb{R}}$

3. (20 points) Consider the equation

20

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}}, \quad \text{for } x > 0$$

$\rightarrow y'' + \frac{y'}{x} + \frac{(x^2-\frac{1}{4})}{x^2}y = \frac{x^{\frac{3}{2}}}{x^2}$

$y'' + \frac{y'}{x} + \frac{(x^2-\frac{1}{4})}{x^2}y = x^{-\frac{1}{2}}$

We are told that the functions $y_1 = x^{-\frac{1}{2}}\sin(x)$ and $y_2 = x^{-\frac{1}{2}}\cos(x)$ are linearly independent solutions of the associated homogeneous equation.

(a) Find the general solution of the given equation.

(b) Find a particular solution to the equation satisfying the initial value conditions $y(\pi/2) = 0$, $y'(\pi) = 0$.

(c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

a) Use variation of parameters to find y_p.

$W_{y_1,y_2} = \det\begin{pmatrix} x^{-\frac{1}{2}}\sin x & x^{-\frac{1}{2}}\cos x \\ -\frac{1}{2}x^{-\frac{3}{2}}\sin x + x^{-\frac{1}{2}}\cos x & -\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x \end{pmatrix} = (-\frac{1}{2}x^{-2}\cos x\sin x - x^{-1}\sin^2 x) - (-\frac{1}{2}x^{-2}\sin x\cos x + x^{-1}\cos^2 x)$

$= \cancel{-\frac{1}{2}x^{-2}\cos x\sin x} - x^{-1}\sin^2 x + \cancel{\frac{1}{2}x^{-2}\sin x\cos x} - x^{-1}\cos^2 x = -x^{-1}(\sin^2 x + \cos^2 x)$

$= -x^{-1} = -\frac{1}{x}$

$v_1 = \int -\frac{f(t)}{W(t)} \cdot y_2\, dx = \int -\frac{x^{-\frac{1}{2}}}{-x^{-1}} \cdot x^{-\frac{1}{2}}\cos(x)\, dx = \int x^{-\frac{1}{2}} \cdot x^{-\frac{1}{2}} \cdot x \cos(x)\, dx = \int \cos(x) dx$

$= \sin(x)$

$v_2 = \int \frac{f(t)}{W(t)}\, y_1\, dx = \int \frac{x^{-\frac{1}{2}}}{-x^{-1}} \cdot x^{-\frac{1}{2}}\sin x\, dx = \int -x^{-\frac{1}{2}} \cdot x^{-\frac{1}{2}} \cdot x^1 \sin x\, dx = \int -\sin x\, dx$

$= \cos(x)$

$y_p = v_1 y_1 + v_2 y_2 = \sin(x) \cdot x^{-\frac{1}{2}}\sin(x) + \cos(x) \cdot x^{-\frac{1}{2}}\cos(x)$

$y_p = x^{-\frac{1}{2}}\sin^2 x + x^{-\frac{1}{2}}\cos^2 x = x^{-\frac{1}{2}}(\sin^2 x + \cos^2 x) = x^{-\frac{1}{2}}$

$y = y_H + y_p$

$\boxed{y(x) = c_1 x^{-\frac{1}{2}}\sin(x) + c_2 x^{-\frac{1}{2}}\cos(x) + x^{-\frac{1}{2}}}$, c_1 and c_2 arbitrary constants $\in \mathbb{R}$

b) $y(\frac{\pi}{2}) = c_1(\frac{\pi}{2})^{-\frac{1}{2}} \cdot \sin(\frac{\pi}{2}) + c_2(\frac{\pi}{2})^{-\frac{1}{2}}\cos(\frac{\pi}{2}) + (\frac{\pi}{2})^{-\frac{1}{2}}$

$= c_1(\frac{\pi}{2})^{-\frac{1}{2}} \cdot 1 + 0 + (\frac{\pi}{2})^{-\frac{1}{2}} = 0 \rightarrow (\frac{\pi}{2})^{-\frac{1}{2}}(c_1 + 1) = 0$

$c_1 + 1 = 0 \rightarrow c_1 = -1$

$y'(x) = -\frac{1}{2}c_1 x^{-\frac{3}{2}}\sin(x) + c_1 x^{-\frac{1}{2}}\cos(x) - \frac{1}{2}c_2 x^{-\frac{3}{2}}\cos(x) - c_2 x^{-\frac{1}{2}}\sin(x) - \frac{1}{2}x^{-\frac{3}{2}}$

$y'(\pi) = 0 + c_1(\pi)^{-\frac{1}{2}}(-1) - \frac{1}{2}c_2(\pi)^{-\frac{3}{2}}(-1) - 0 - \frac{1}{2}(\pi)^{-\frac{3}{2}} \rightarrow$

*negative exponent derivative careful

$$y'(\pi) = 0 = -C_1(\pi)^{-\frac{1}{2}} + \tfrac{1}{2}C_2(\pi)^{-\frac{3}{2}} - \tfrac{1}{2}(\pi)^{-\frac{3}{2}}\Big)\pi^{\frac{1}{2}}$$

$$= 0 = -C_1 + \tfrac{1}{2}C_2(\pi)^{-1} - \tfrac{1}{2}(\pi)^{-1}$$

and from before, $C_1 = -1$ so $0 = -(-1) + \frac{1}{2}\cdot\frac{C_2}{\pi} - \frac{1}{2\pi}$

$$\left(0 = 1 + \frac{C_2}{2\pi} - \frac{1}{2\pi}\right)2\pi \rightarrow 0 = 2\pi + C_2 - 1$$

so $C_2 = 1 - 2\pi$

Check: $0 = -(-1) + \frac{1}{2}(1-2\pi)\cdot\frac{1}{\pi} - \frac{1}{2\pi}$

$0 = 1 + \frac{1}{2\pi} - 1 - \frac{1}{2\pi} = 0$ ✓

$$\boxed{y(t) = (-1)x^{-\frac{1}{2}}\sin(x) + (1-2\pi)x^{-\frac{1}{2}}\cos(x) + x^{-\frac{1}{2}}}$$

is the solution to the IVP

C) No, the Uniqueness and Existence Theorem cannot be applied here for 2nd order linear equations because this theorem states that for $y'' + p(t)\cdot y' + q(t)\cdot y = f(t)$ Then $p(t), q(t)$, and $f(t)$ must be continuous on the interval containing t_0 and satisfying boundary conditions $y(t_0) = \alpha$, $y'(t_0) = \beta$ in order for there to exist a unique solution to the differential equation on the interval.

However, in this problem, the boundary conditions $y(\frac{\pi}{2}) = 0$ and $y'(\pi) = 0$ are given at two different points $t_0 = \frac{\pi}{2}$, $t_0 = \pi$ and since $\frac{\pi}{2} \neq \pi$ the uniqueness and existence theorem cannot apply here.

11

4. (14 points)

(a) Write the **definition** of what it means for three functions $y_1(t), y_2(t), y_3(t)$ to be linearly independent.

(b) Let y_1, y_2 be solutions to the innhomogeneous linear ODE $y'' - y = f(x)$. We are told that $y_1(0) = 0, y_1'(0) = 2, y_2(0) = 2, y_2'(0) = 2$, and $y_1(1) = 1$. Find $y_2(1)$. Justify your answer.

a) $y_1(t), y_2(t), y_3(t)$ are linearly independent if the linear combination

$c_1y_1(t) + c_2y_2(t) + c_3y_3(t) = 0$ with constants $c_1, c_2, c_3 \in \mathbb{R}$ has $c_1 = c_2 = c_3 = 0$ so that y_1, y_2, and y_3 have only the trivial relation where $c_1 = c_2 = c_3 = 0$. This means also that the $W_{y_1,y_2,y_3}(t) = \det\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \neq 0$.

9

b) ~~$W_{y_1,y_2}(t) = W_{y_1,y_2}(0) \cdot e^{-\int p(t)dt}$ where $p(t) = 0$ here.~~

~~$y_1y_2' - y_2y_1'$~~ $W_{y_1,y_2}(0) = \det\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = \det\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$

$= 0 - 4 = -4$

~~$W_{y_1,y_2}(t) = -4 \cdot e^{-\int p(t)dt}$~~

~~At $t=1$, $y_1y_2' - y_2y_1' = -4 \cdot e^{-\int p(t)dt}$~~

~~$y_2' - y_2y_1' = -4e^{\int p(t)dt}$~~

$W_{y_1,y_2}(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

$= y_1y_2' - y_2 \cdot y_1'$

At $t = 1$ is $y_2' - y_2 \cdot y_1'$ 2

Variation of parameters.

$V_1 = \int -\frac{f(t)}{w(t)} \cdot y_2 \, dt$

$= \int -\frac{f(x)}{}$

$y_p = v_1y_1 + v_2y_2$

homogeneous solution: $t^2 - 1 = (t+1)(t-1)$

so $y_H(t) = c_1e^t + c_2e^{-t}$

$y(t) = c_1e^t + c_2e^{-t} + \int \frac{-f(t)}{w(t)} \cdot y_2 dt \cdot y_1 + \int \frac{f(t)}{w(t)} \cdot y_1 dt \cdot y_2$

4b) Idea: Because y_1 & y_2 are particular solutions, $y = y_2 - y_1$

Solves the homogeneous equation

$y'' - y = 0$

Solves the homogeneous equation

$p(\lambda) = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$

so $y_H(t) = C_1 e^t + C_2 e^{-t}$

so $y = y_H + y_p$ where $y_p = y_1$ and $y = y_2$

so $y = C_1 e^t + C_2 e^{-t} + y_1$ or $y_2 = C_1 e^t + C_2 e^{-t} + y_1$

$y_1(0) = 0$ & $y_2(0) = 2 \Rightarrow 2 = C_1 + C_2$

$y_2' = C_1 e^t - C_2 e^{-t} + y_1'$

$y_1'(0) = 2$ & $y_2'(0) = 2 \Rightarrow 2 = C_1 - C_2 + 2$

$0 = C_1 - C_2$

Solve $\begin{cases} C_1 + C_2 = 2 \\ C_1 - C_2 = 0 \end{cases} \longrightarrow 2C_1 = 2$

$C_1 = 1$

and $C_2 = 1$

so $y_2 = C_1 e^t + C_2 e^{-t} + y_1$

$y_2 = e^t + e^{-t} + y_1$

$y_2(1) = e^1 + e^{-1} + y_1(1)$

$$\boxed{y_2(1) = e + \frac{1}{e} + 1}$$

★ applies for any linear equation ★

$y_2'' - y_2 = f(x)$

$-\quad y_1'' - y_1 = f(x)$

$(y_2 - y_1)'' - (y_2 - y_1) = 0$

$y'' - y = 0$

so $y = y_2 - y_1$

$\uparrow$

y_H so

$y_2 = y_H + y_1$

$y_2 = y_1 + y_p$

$y = y_H + y_p$

$\uparrow$

$y_2 = y_H + y_1$ is allowed

For any linear diff eq, subtracting 2 particular solutions results in the homog eq.