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Math 33B, Lecture 2
Spring 2018
05/21/18
Time Limit: 50 Minutes

Name (Print): Ben Limpanukorn
SID Number:
Signature:

Day \ T.A.	Blaine	Frank	Siting
Tuesday	2A	2C	2E
Thursday	2B	2D	~~2F~~

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and cross the box corresponding to your discussion section.
2. Use a PEN to record your final answers.
3. If you need more space, use the back of this page and pages 6,8.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	18	18
2	18	18
3	20	20
4	14	11
Total:	70	67

1. (18 points) Consider the following differential equation

$$y'' - 2y' + 2y = e^t \sin(t) \quad = \operatorname{Im}/ e^t e^{it} = e^{(1+i)t}$$

(a) Find the general solution to the associated homogeneous equation.

(b) Find the general solution to the given inhomogeneous equation.

a) homo. char: $x^2 - 2x + 2 = 0$

$(x^2 - 2x + 1) = -2 + 1$

$(x-1)^2 = -1$

$x = 1 \pm i$

$z_1 = e^{(1+i)t}$ $z_2 = e^{(1-i)t} = e^t(\cos(t) - i\sin(t))$

$\boxed{y_h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)}$ 9

b) $y'' - 2y' + 2y = e^{(1+i)t}$

guess: $y_p(t) = v_0 t e^{(1+i)t}$

$y_p'(t) = v_0(1+i)te^{(1+i)t} + v_0 e^{(1+i)t}$

$= v_0 e^{(1+i)t}((1+i)t + 1)$

$y_p''(t) = (1+i)v_0((1+i)t+1)e^{(1+i)t} + (1+i)v_0 e^{(1+i)t}$

$= v_0 e^{(1+i)t}((1+i)((1+i)t+1) + 1 + i)$

$= v_0 e^{(1+i)t}(\cancel{t} + it + 1 + it - \cancel{t} + i + 1 + i)$

$= v_0 e^{(1+i)t}(2 + 2it + 2i)$

sub: $(v_0 e^{(1+i)t}(2 + 2it + 2i))$

$-2(v_0 e^{(1+i)t}(t + it + 1))$

$+2(v_0 t e^{(1+i)t}) = e^{(1+i)t}$

$2v_0 e^{(1+i)t} + \cancel{2iv_0 t e^{(1+i)t}} + 2iv_0 e^{(1+i)t}$

$\cancel{-2v_0 t e^{(1+i)t}} \cancel{- 2v_0 it e^{(1+i)t}} - 2v_0 e^{(1+i)t}$

$\cancel{+2v_0 t e^{(1+i)t}} = e^{(1+i)t}$

$2iv_0 e^{(1+i)t} = e^{(1+i)t}$

$v_0 = -\frac{i}{2}$ $\quad y_p(t) = -\frac{i}{2} t e^{(1+i)t}$

$= -\frac{it}{2} e^t(\cos(t) + i\sin(t))$

$= \underbrace{-\frac{it e^t \cos(t)}{2}} + \frac{t}{2} e^t \sin(t)$

$$\boxed{y(t) = \underbrace{-\frac{t e^t \cos(t)}{2}}_{\text{ptic}} \underbrace{+ c_1 e^t \cos(t) + c_2 e^t \sin(t)}_{\text{homo}}}$$

2. (18 points) Consider the following differential equation

$$t^2y'' - 3ty' + 4y = 0 \quad \text{solve for } t > 0$$

(a) Find two linearly independent solutions of the equation. Show that the solutions are linearly independent.

(b) What is the general solution of the equation. Justify your answer.

a) $y = t^\lambda \rightarrow y' = \lambda t^{\lambda-1} \quad y'' = \lambda(\lambda-1)t^{\lambda-2}$

$t^2(\lambda(\lambda-1)t^{\lambda-2}) - 3t(\lambda t^{\lambda-1}) + 4(t^\lambda) = 0$

$t^\lambda(\lambda^2 - \lambda - 3\lambda + 4) = 0$

$t^\lambda(\lambda^2 - 4\lambda + 4) = 0$

$t^\lambda(\lambda-2)^2 = 0$

$\boxed{y_1(t) = t^2 \quad y_2(t) = (\ln t)t^2}$

$y' = 2t\ln t + t$

$y'' = \frac{2t}{t} + 2\ln t + 1 = 3 + 2\ln t$

lin. independent: $W_{y_1, y_2}(t) = \begin{vmatrix} t^2 & t^2\ln t \\ 2t & 2t\ln t + \frac{t^2}{t} \end{vmatrix}$

$= t^2(2t\ln t + t) - 2t^3\ln t$

$= \cancel{2t^3\ln t} + t^3 - \cancel{2t^3\ln t}$

$= t^3 \neq 0$ | const. zero func.

$\Longrightarrow$ lin. ind. b/c $W_{y_1, y_2} \neq 0$

b) General sol: $\boxed{y(t) = C_1t^2 + C_2t^2\ln t} = C_1y_1 + C_2y_2$

proof: $y_1(t)$ is a sol: $t^2(2) - 3t(2t) + 4(t^2) = 0$

$y_1 = t^2$

$y_1' = 2t$ $\quad \cancel{2t^2} - \cancel{6t^2} + \cancel{4t^2} = 0$

$y_1'' = 2$ $\quad 0 = 0$

✓ a solution by substitution

$y_2(t)$ is a sol: $t^2(3 + 2\ln t) - 3t(2t\ln t + t) + 4(t^2\ln t) = 0$

$\cancel{3t^2} + \cancel{2t^2\ln t} - \cancel{6t^2\ln t} - \cancel{3t^2} + \cancel{4t^2\ln t} = 0$

$0 = 0$

$y(t) = C_1t^2 + C_2t^2\ln t$ is a sol. ✓ a sol. by sub.

Since the ODE is linear and $y(t)$ is a linear combination of solutions $y_1\ y_2$ (linearity property)

The general solution spans all possible solutions since $y_1\ y_2$ are linearly independent solutions of the eq.

3. (20 points) Consider the equation

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{\frac{3}{2}}, \quad \text{for } x > 0$$

$\frac{x^{3/2}}{x^2} = x^{-1/2}$

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We are told that the functions $y_1 = x^{-\frac{1}{2}}\sin(x)$ and $y_2 = x^{-\frac{1}{2}}\cos(x)$ are linearly independent solutions of the associated homogeneous equation.

(a) Find the general solution of the given equation.

(b) Find a particular solution to the equation satisfying the initial value conditions $y(\pi/2) = 0$, $y'(\pi) = 0$.

(c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

a) $y_p(t) = v_1 y_1 + v_2 y_2$

$$W_{y_1 y_2} = \begin{pmatrix} x^{-1/2}\sin x & x^{-1/2}\cos x \\ x^{-1/2}\cos t - \frac{1}{2}x^{-3/2}\sin t & -x^{-1/2}\sin x - \frac{1}{2}x^{-3/2}\cos x \end{pmatrix}$$

$-x^{-1}\sin^2 x - \cancel{\frac{1}{2}x^{-2}\sin x\cos x}$

$-\left(x^{-1}\cos^2 x - \cancel{\frac{1}{2}x^{-2}\sin x\cos x}\right)$

$-1\left(x^{-1}(\sin^2 x + \cos^2 x)\right)$

$= -\frac{1}{x} = -x^{-1}$

$v_1 = -\int \frac{f(t)}{W(t)} \cdot y_2(t)\,dt$

$= -\int \frac{x^{-1/2}}{-x^{-1}}(x^{-1/2}\cos(x))\,dx$

$= \int x^{1/2}x^{-1/2}\cos(x)\,dx$

$= \int \cos(x)\,dx = \sin x + c$

$v_2 = \int \frac{f(t)}{W(t)} \cdot y_1(t)\,dt$

$= \int \frac{x^{-1/2}}{-x^{-1}}(x^{-1/2}\sin(x))\,dx$

$= \int -x^{1/2}x^{-1/2}\sin x\,dx$

$= \int -\sin x\,dx = \cos x + c$

$y(t) = (\sin x + c_1)x^{-1/2}\sin(x) + (\cos x + c_2)x^{-1/2}\cos(x)$

$$\boxed{y(t) = x^{-1/2} + c_1 x^{-1/2}\sin x + c_2 x^{-1/2}\cos x}$$

b) $y'(t) = -\frac{1}{2}x^{-3/2} + c_1\left(x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x\right) + c_2\left(-x^{-1/2}\sin x - \frac{1}{2}x^{-3/2}\cos x\right)$

$0 = -\frac{1}{2}\pi^{-3/2} + c_1\left(\pi^{-1/2}\cos\pi - \cancel{\frac{1}{2}\pi^{-3/2}\sin\pi}\right) + c_2\left(0 - \frac{1}{2}\pi^{-3/2}\cos\pi\right)$

$0 = -\frac{\pi^{-3/2}}{2} - c_1\pi^{-1/2} + c_2\frac{\pi^{-3/2}}{2}$

$0 = \left(\frac{\pi}{2}\right)^{-1/2} + c_1\left(\frac{\pi}{2}\right)^{-1/2} + 0$

$c_1 = -1$

$0 = -\frac{\pi^{-3/2}}{2} + \pi^{-1/2} \cdot \frac{2}{2} + c_2\frac{\pi^{-3/2}}{2}$

$\frac{\cancel{2}}{\pi^{-3/2}}\left(\frac{-2\pi^{-1/2} + \pi^{-3/2}}{\cancel{2}}\right) = c_2$

$-2\pi + 1 = c_2$

$$\boxed{y(t) = x^{-1/2} - x^{-1/2}\sin x + (1 - 2\pi)x^{-1/2}\cos x}$$

c on back ↓

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c) No because the theorem only applies to IVP with the same initial value point (same t val not neccessarily same ~~ar~~ value)

i.e. $y(t_0) = \alpha \quad y'(t_0) = \beta$

must match

However, the given IVP ~~an~~ has the points $y(\frac{\pi}{2}) = 0 \quad y'(\pi) = 0$

$\frac{\pi}{2} = t_0 \neq \pi = t_1$

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4. (14 points)

(a) Write the definition of what it means for three functions $y_1(t), y_2(t), y_3(t)$ to be linearly independent.

(b) Let y_1, y_2 be solutions to the innhomogeneous linear ODE $y'' - y = f(x)$. We are told that $y_1(0) = 0, y_1'(0) = 2, y_2(0) = 2, y_2'(0) = 2$, and $y_1(1) = 1$. Find $y_2(1)$. Justify your answer.

a) $y_1\ y_2\ y_3$ are linearly independent if the only combination of values for $c_1\ c_2\ c_3$ of $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0$ is the trivial set : $c_1 = c_2 = c_3 = 0$

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b) $y_1(0) = 0$ $\quad y_2(0) = 2$ $\quad y'' - y = f(x)$ $\quad x^2 - 1 = 0$

$y_1'(0) = 2$ $\quad y_2'(0) = 2$ $\quad y'' - y = 0$ $\quad x^2 = 1$

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$y(1) = 1$ $\quad x = \pm 1$

$y_1 + y_2$ is a sol of th

$y = \cancel{e^t}$ $\quad y = \cancel{e^{-t}}$

$y_1 - y_2$ solves the homogeneous equation

$y' = et$ $\quad y' = -e^{-t}$

$\boxed{y_2(1) = 2e^1 = 2e}$

$y_1 = c_1 e^t + c_2 e^{-t}$

$0 = c_1 + c_2$

$2 = c_1 - c_2$

Since $y_2(t) = 2et$

$2 = 2c_1$

$c_1 = 1 \quad c_2 = -1$

$y_2 = c_1 e^t + c_2 e^{-t}$

$2 = c_1 + c_2$

$2 = c_1 - c_2$

$4 = 2c_1$

$c_1 = 2 \quad c_2 = 0$