4. (18 points) Consider the following differential equation

$$y'' + 2y' + y = \frac{e^{-x}}{x^2 - 1} \quad \text{for } -1 < x < 1$$

13/18

(a) Find the general solution to the associated homogeneous equation.
(b) Find the general solution to the given inhomogeneous equation.

a

$y'' + 2y' + y = 0$

$y = e^{\lambda t}$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1$

$\cancel{y_h = c_1 e^{-t} + c_2 t e^{-t}}$

$\boxed{y_h = c_1 e^{-x} + c_2 x e^{-x}}$

$\cancel{y_h = c_1 e^{-t} + c_2 t e^{-t}}$

b.

$W = \det \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{pmatrix} = e^{-2t}(-t + 1 + t) = e^{-2x}$

$y = v_1 y_1 + v_2 y_2$

$v_1 = -\int \frac{f}{w} y_2 \, dx = -\int \frac{e^x}{x^2-1} e^{-x} \, dx$

$v_1 = -\int \frac{1}{x^2-1} \, dx$ $\quad \frac{1}{1-x^2}$

v_2

$v_1 = \tan^{-1} x$

$v_2 = \int \frac{f}{w} y_1 \, dx \quad v_2 = \int \frac{e^x}{x^2-1} x e^{-x} \, dx$

$v_2 = \int \frac{x}{x^2-1} dx \quad v_2 = \frac{1}{2} \ln|(x^2-1)|$

$y_p = e^{-x} \tan^{-1}(x) + \frac{1}{2} x e^{-x} \ln|x^2-1|$

$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + e^{-x} \tan^{-1}(x) + \frac{1}{2} x e^{-x} \ln|x^2-1|}$

1. (20 points)
   (a) Find the general solution to the following system
   $$\vec{y}' = \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix} \vec{y}$$
   (b) Are the functions $f_1 = t - 1$, $f_2 = t^2 + 1$, and $f_3 = t^2 + t$ linearly independent? Justify your answer.

a. $y' = \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix} y$

10

$\det\begin{pmatrix} 2-\lambda & 5 \\ 2 & -1-\lambda \end{pmatrix} = (\lambda-2)(\lambda+1) - 10$

$\lambda^2 - \lambda - 12$

$(\lambda-4)(\lambda+3)$

$\lambda = 4, -3$

$\lambda = 4, \begin{pmatrix} -2 & 5 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 = \begin{pmatrix} 5a \\ 2a \end{pmatrix}$

$\lambda = -3 \begin{pmatrix} 5 & 5 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_2 \begin{pmatrix} a \\ -a \end{pmatrix}$

$y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$\cancel{y = c_1 e^{4t}\begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$$\boxed{y = c_1 e^{4t}\begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

b.

10

$W_{f_1, f_2, f_3} = \det\begin{pmatrix} t-1 & t^2+1 & t^2+t \\ 1 & 2t & 2t+1 \\ 0 & 2 & 2 \end{pmatrix}$

$(t-1)(4t - (4t+2)) + (t^2+1)(-2) + (t^2+t)(2)$

$-2t + 2 + -2t^2 - 2 + 2t^2 + 2t = 0$

$W_{f_1, f_2, f_3} = 0$

No f_1, f_2, and f_3 are not linearly independent

[illegible] (linear combination)

2. (18 points) Consider the differential equation

$$y'' + 4y = \sin(x)$$

(a) Find the general solution to the equation.

(b) Find a particular solution to the equation satisfying the initial value conditions $y(0) = \frac{1}{2}$, $y'(\pi) = 0$.

(c) Is it possible to apply the Uniquenss and Existence Theorem for second-order linear equations to the initial value problem in part (b)? Justify your answer.

a. $y'' + 4y = \sin x$ $\qquad y' = a\cos x - b\sin x$

4

Guess $y = a\sin x + b\cos x$ $\qquad y'' = -a\sin x - b\cos x$

$-a\sin x - b\cos x + 4(a\sin x + b\cos x) = \sin x$

$3a\sin x + 3b\cos x = \sin x$

$y_h = y'' + 4y = 0$ $\qquad a = \frac{1}{3},\ b = 0$ $\qquad y_p = \frac{1}{3}\sin x$

$y = y_h + y_p$

Guess $y = a\sin x + b\cos x$

$y = \sin 2x,\ \cos 2x$

$$\boxed{y = c_1\sin(2x) + c_2\cos(2x) + \frac{1}{3}\sin(x)}$$

$y' = +2\cos 2x$

$y'' = -4\sin 2x$

$y_h = c_1\sin 2x + c_2\cos 2x$

$y = \sin 2x + \cos 2x + \frac{1}{3}\sin x$

$y' = 2\cos 2x - 4\sin 2x + \frac{1}{3}\cos x$

$y'' = -4\sin 2x - 8\cos 2x - \frac{1}{3}\sin x$

3

b. $y = c_1\sin 2x + c_2\cos 2x + \frac{1}{3}\sin x$

$\frac{1}{2} = c_2$

$y' = 2c_1\cos 2x - 2c_2\sin 2x + \frac{1}{3}\cos x$ $\qquad -\frac{1}{3}\cos 2x + \frac{1}{3}\cos x$

$0 = 2c_1 + \frac{1}{3}$

$c_1 = -\frac{1}{6}$

$$\boxed{y = -\frac{1}{6}\sin(2x) + \frac{1}{2}\cos(2x) + \frac{1}{3}\sin(x)}$$

4

c. $\boxed{\text{No.}}$ Uniqueness + Existence Theorem does not apply for part b because $y(t_0) = \alpha$, $y_1(t_0) = \beta$ must be for the same value of t_0. Since the values for y and y' are given for pts $x = 0$ and π, we cannot use the theorem because they're not for the same point.

3. (14 points) Find the general solution to the following equation.

$$t^2y'' + 2ty' = (t-1) \quad \text{for } t > 0$$

$t^2y'' + 2ty' = t-1$

y_h, $t^2y'' + 2ty' = 0$

$y = t^r$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$r(r-1)t^r + 2rt^r = 0$

$r^2 - r + 2r = 0$

$r^2 + r = 0$

$r = 0, -1$

$y_h = c_1 + c_2 t^{-1}$

$1 + 2t^{-1}$

$-2t^{-2}$

$4t^{-3}$

$y_1 = 1 \quad y_2 = t^{-1}$

$y = y_1v_1 + y_2v_2$

$v_1 = -\int \frac{f}{w} y_2 \, dt$

$v_2 = \int \frac{f}{w} y_1 \, dt$

$W = \begin{pmatrix} 1 & t^{-1} \\ 0 & -t^{-2} \end{pmatrix} = -t^{-2}$

$v_1 = -\int \frac{t-1}{-t^{-2}} t^{-1} \, dt$

$v_1 = \int t^2 - t \, dt = \frac{1}{3}t^3 - \frac{1}{2}t^2$

$v_2 = \int \frac{t-1}{-t^{-2}} \, dt = \int -t^3 + t^2 \, dt = -\frac{1}{4}t^4 + \frac{1}{3}t^3$

$y_p = \frac{1}{3}t^3 - \frac{1}{2}t^2 + -\frac{1}{4}t^3 + \frac{1}{3}t^2$

$y_p = \frac{1}{12}t^3 - \frac{1}{6}t^2$

$y' = \frac{1}{4}t^2 - \frac{1}{3}t$

$y'' = \frac{1}{2}t - \frac{1}{3}$

$\frac{1}{2}t^3 - \frac{2}{3}t^2$

$+ \frac{1}{2}t^3 - \frac{1}{3}t^2$

$t^3 \quad t^2$

$-a + 2a + 2bt$

$2bt + a$

$= t - 1$

y_p: $t^2y'' + 2ty' = t-1$

$a = -1$

$b = \frac{1}{2}$

$y' = \frac{a}{t} + b$

$y'' = \frac{-a}{t^2}$

$y = a\ln t + bt$

y': $y' = \frac{a}{t} + b$

$y'' = \frac{-a}{t^2}$

$2ty'$

$2a + 2bt$

$-a$

$-a + a + bt$

$y = at + b$

$y' = a$

$y'' = 0$

$2bt - a$

$= t - 1$

$2t + a = t - 1$

$a = 1$

$b = \frac{1}{2}$

$y_h = -\ln|t| + \frac{1}{2}t$

~~$y = c_1 + c_2t^{-1} + -\ln|t|$~~ $+ \frac{1}{2}t$

$$\boxed{y = c_1 + c_2t^{-1} - \ln|t| + \frac{1}{2}t}$$

14