Math 33B, Lecture 2
Fall 2016
10/17/16
Time Limit: 50 Minutes

Name (Print): ______________________
SID Number: ______________________

Signature: ______________________

Day \ T.A.	Yanli	Ben	Madeleine
Tuesday	**2A**	**2C**	**2E**
Thursday	**2B**	**2D**	**2F**

This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

Instructions

1. Enter your name, SID number, and signature on the top of this page and **cross the box** corresponding to your discussion section.
2. **Use a pen** to record your final answers.
3. Use the back of this pages and the last page if you need more space.
4. Calculators, computers, books or notes of any kind are not allowed.
5. Show your work. Unsupported answers will not receive full credit.
6. Good Luck!

Problem	Points	Score
1	15	15
2	15	15
3	15	15
4	15	13
5	15	9
Total:	75	67

1. (15 points) Consider the differential equation

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$$a(x^2 + y^2)dx + (xy^n + y^2)dy = 0 \qquad \left(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\right)dx + (xy+y^2)dy$$

(a) Find values a, n such that the equation is exact.

(b) Find the general solution to the equation with the values a, n you found in part (a).

(a) : $\frac{d}{dy} P(x,y) = \frac{d}{dy} a(x^2+y^2) = 2ay$ to be exact, $2ay = y^n \Rightarrow \boxed{\begin{cases} a = \frac{1}{2} \\ n = 1 \end{cases}}$

$\frac{d}{dx} Q(x,y) = \frac{d}{dx}(xy^n + y^2) = y^n \Rightarrow \begin{cases} 2a = 1 \\ n = 1 \end{cases}$

(b) $\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right)dx + (xy+y^2)\,dy = 0$

$F(x,y) = \int xy + y^2\, dy = \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \phi(x)$

$\frac{d}{dx} F(x,y) = \frac{1}{2}y^2 + \phi'(x)$, $\phi'(x) = \frac{1}{2}x^2 \Rightarrow \phi(x) = \frac{1}{6}x^3$

General solution : $\boxed{F(x,y) = \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \frac{1}{6}x^3 = C}$

2. (15 points) A tank is filled with 330 gallons of pure water. Solution containing 2 lb of salt per gallon is pumped into the tank at the rate of 4 gal/min. At the same time, the solution in the tank is pumped out at the same rate. Let $x(t)$ be the number of pounds of salt in the tank at time t.

15

(a) Use the information to write an initial value problem which is satisfied by $x(t)$.

(b) Find $x(t)$.

(a) $X'(t) = 2\cdot 4 - \frac{4}{330}\cdot X(t)$, $X(0)=0$

Find the solution for the differential Equation $X'(t) = 8 - \frac{4}{330}\cdot X(t)$

with initial value $X(0) = 0$

(b) $\frac{dX}{dt} = \frac{2640-4X}{330} \Rightarrow \frac{1}{2640-4X}\,dX = \frac{1}{330}\,dt$

$\Rightarrow -\frac{1}{4}\ln|2640-4X| = \frac{t}{330} + C$

$\ln|2640-4X| = -\frac{2}{165}t + C$

$2640-4X = A\cdot e^{-\frac{2}{165}t}$

$X = 660 - A\cdot e^{-\frac{2}{165}t}$

$X(0)=0 \Rightarrow A = 660$

So $X(t) = 660 - 660\cdot e^{-\frac{2}{165}t}$

$$\boxed{X(t) = 660 - 660\cdot e^{-\frac{2}{165}t}}$$

165
2)330
2
13
12
10

1320
660

3. (15 points) Consider the differential equation $\frac{dx}{dt} = \frac{t^{\frac{1}{3}} - x}{t+2}$

$$(t+2)\frac{dx}{dt} + x = t^{1/3}$$

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(a) Use the variation of parameters method to solve the equation with the initial value condition $x(0) = 0$.

(b) What is the largest rectangle R in the tx-plane containing the point $(0,0)$ to which we can apply the existence and uniqueness Theorem? Justify your answer.

(a) $\frac{dx}{dt} = \frac{t^{\frac{1}{3}} - x}{t+2} = -\frac{x}{t+2} + \frac{t^{\frac{1}{3}}}{t+2}$

$\frac{1}{x}dx = -\frac{1}{t+2}dt$

$\ln|x| = -\ln|t+2| + C$

$x = A\frac{1}{t+2}$ $\quad x_h = \frac{1}{t+2}$ $\quad v' \cdot \frac{1}{t+2} = \frac{t^{\frac{1}{3}}}{t+2}$

$v' = t^{\frac{1}{3}}$

$v = \frac{3}{4}t^{\frac{4}{3}} + C$

$x(t) = \left(\frac{3}{4}t^{\frac{4}{3}} + C\right) \cdot \frac{1}{t+2}$

$x(0) = 0 \Rightarrow \frac{C}{2} = 0 \Rightarrow C = 0$

$\Rightarrow \boxed{x(t) = \left(\frac{3}{4}t^{\frac{4}{3}}\right) \cdot \frac{1}{t+2}}$

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(b) $f(t,x) = \frac{t^{\frac{1}{3}} - x}{t+2}$ $\quad \frac{df}{dx} = -\frac{1}{t+2}$

They are not continuous when $t = -2$. So the largest R is

$R = \boxed{(-2, \infty) \times (-\infty, \infty)}$

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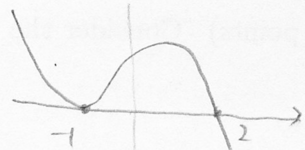
4. (15 points) Consider the differential equation



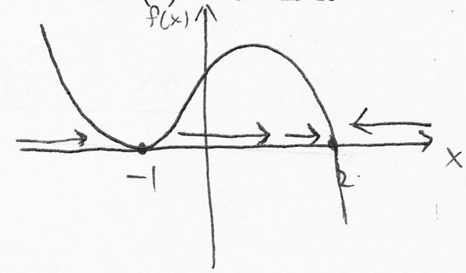
$$\frac{dx}{dt} = (x+1)^2(2-x)$$

(a) Identify the equilibrium points and draw the phase line diagram of the equation.

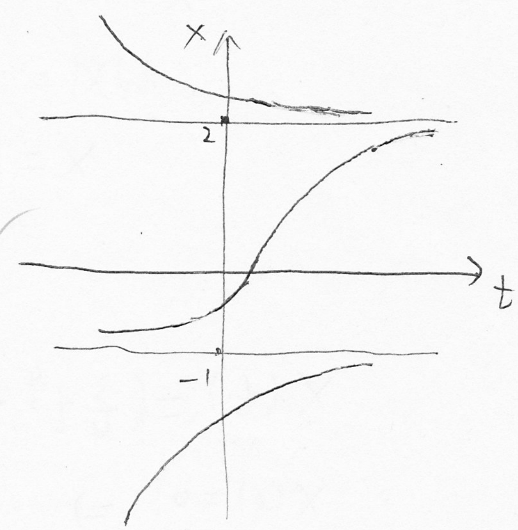
(b) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points.

(c) Let $x(t)$ be a particular solution to the equation which satisfies $x(1) = -3$. Is it possible that $x(2) = 0$? Justify your answer.

X(1) = -3



(a) equilibrium points : X = 2 and X = -1



(b) stable points : X = 2

unstable points : X = -1

(c) It is impossible that X(2) = 0 if X(1) = -3

~~According to the uniqueness theorem.~~

~~Because~~

Because if X(1) = -3, then we have $\lim_{t \to \infty} X(t) = -1$.

According to the uniqueness theorem, the solution cannot cross the equilibrium solution X = -1. So X(2) can never be 0 because X(2) < -1.

continuous?

Answer:

Since f(t, X) and $\frac{d}{dt}$f(t, X) are continuous, so the theorem of existence and uniqueness applies

So if $\begin{cases} X(1) = -3 \\ X(2) = 0 \end{cases}$ then the solution X must intersect the solution X = -1 somewhere between t=1, 2

Contradiction

5. (15 points) Consider the differential equation.

$$3y' = y^4(3t^2 - 4t)$$

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1

(a) Find the solution to this equation with the initial value condition $y(1) = 1$ and determine its interval of existence.

(b) Find all values $\mathbf{y_0}$ for which $(-\infty, \infty)$ is the interval of existence of the solution to the equation with the initial value condition $y(0) = \mathbf{y_0}$. Justify your answer.

(a) $3\frac{dy}{dt} = y^4(3t^2 - 4t)$

when $y \neq 0$ $\quad y^{-4}\,dy = (t^2 - \frac{4}{3}t)\,dt$

$$-\frac{1}{3}y^{-3} = \frac{1}{3}t^3 - \frac{2}{3}t^2 + C$$

$$y^{-3} = -t^3 + 2t^2 + C$$

Given $y(1) = 1$, $\quad 1 = -1 + 2 + C \Rightarrow C = 0.$

$$\Rightarrow y^{-3} = -t^3 + 2t^2$$

$$y = (-t^3 + 2t^2)^{-\frac{1}{3}}$$

when $y = 0$, $y = 0$ is also a solution. So $\boxed{y = (-t^3+2t^2)^{-\frac{1}{3}} \text{ or } y = 0}$

$y = \frac{1}{\sqrt[3]{t^2(2-t)}}$, $t^2(2-t) \neq 0 \Rightarrow t \neq 0, t \neq 2$ $\quad \boxed{I = (0, 2)}$

(b) $-t^3 + 2t^2 + C > 0.$ is not always true given some C?

why.

So $\boxed{y_0 = 0.}$

why?

No justification

Extra paper 1

Suppose $x(t)$ is a solution to

$$\begin{cases} \frac{dx}{dt} = x \sin(t^2) \\ x(0) > 0 \end{cases}$$ prove that $x(t) > 0$ for $t \geq 0$

$f(t,x) = x \cdot \sin(t^2)$ $\qquad \frac{d}{dt} f(t,x) = x \cdot 2t \cdot \cos(t^2)$