5. (15 points) Consider the differential equation.

$$3y' = y^4(3x^2 + 4x)$$

with the initial value condition $y(-1) = -1$ and

8

(a) Find the solution to this equation with the initial value condition $y(-1) = -1$ and determine its interval of existence.

1 P (b) Find all values y_0 for which $(-\infty, \infty)$ is the interval of existence of the solution to the equation with the initial value condition $y(0) = y_0$. Justify your answer.

a) $3y' = y^4(3x^2+4x)$

$\frac{3}{y^4}\,dy = (3x^2+4x)dx$

$\int \frac{3}{y^4}\,dy = \int (3x^2+4x)\,dx$

$-\frac{1}{y^3} = x^3 + 2x^2 + C,$

$y(-1) = -1$

$\frac{-1}{(-1)^3} = -1^3 + 2(-1)^2 + C$

$1 = -1 + 2 + C$

$C = 0$

$\therefore -\frac{1}{y^3} = x^3 + 2x^2$

$\frac{1}{y^3} = -x^3 - 2x^2$

$y^3 = \frac{1}{-x^3-2x^2}$

$y = (-x^3 - 2x^2)^{-\frac{1}{3}}$ $\qquad y = (-x^3 - 2x^2)^{-\frac{1}{3}}$ ✓

To exist, $-x^3 - 2x^2 \neq 0$

$-x^3 - 2x^2 = 0$

$x^2(-x+2) = 0$

$x \neq 0, \pm 2$

$\therefore$ Interval of existence ✓

$I = (-2, 0)$ $\qquad I = (-2, 0)$

b) for I to be $(-\infty, \infty)$

$-x^3 - 2x^2 + y_0 \neq 0$

$y(0) = y_0 = \left(\frac{1}{C}\right)^{\frac{1}{3}} = C^{-\frac{1}{3}}$

P

1. (15 points)
Consider the differential equation

$$(x + xy^n)dx + a(x^2 + y^2)dy = 0$$

15

(a) Find values a, n such that the equation is exact.
(b) Find the general solution to the equation with the values a, n you found in part (a).

a) $P = x + xy^n$ $\qquad Q = a(x^2+y^2)$

For exact, $\frac{dP}{dy} = \frac{dQ}{dx}$.

$\frac{dP}{dy} = nxy^{n-1}$ $\qquad \frac{dQ}{dx} = 2ax$

compare,

$n-1=0,\ n=1$

$n = 2a$

$a = \frac{1}{2}$.

b) $F(x,y) = \int (x + xy)dx$

$= \frac{x^2}{2} + \frac{x^2}{2}y + \phi(y)$

$\frac{d}{dy}\left(\frac{x^2}{2} + \frac{x^2}{2}y + \phi(y)\right) = \frac{1}{2}(x^2+y^2)$

$\frac{x^2}{2} + \phi'(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$\phi'(y) = \frac{1}{2}y^2$

$\phi(y) = \frac{1}{6}y^3 + c$

$\therefore F(x,y) = \frac{x^2}{2} + \frac{x^2}{2}y + \frac{y^3}{6} = C$.

$F(x,y) = \cancel{\frac{x^2}{2}}$

$\frac{x^2}{2} + \frac{x^2}{2}y + \frac{y^3}{6} = C$

2. (15 points) A tank is filled with 210 gallons of pure water. Solution containing 3 lb of salt per gallon is pumped into the tank at the rate of 4 gal/min. At the same time, the solution in the tank is pumped out at the same rate. Let $x(t)$ be the number of pounds of salt in the tank at time t.

(a) Use the information to write an initial value problem which is satisfied by $x(t)$.

(b) Find $x(t)$.

15

a) Rate in $= 3 \times 4 = 12$.

Rate out $= 4 \dfrac{x(t)}{210} = \dfrac{2x(t)}{105}$.

$\cancel{x} : \dfrac{dx}{dt} = 12 - \dfrac{2x}{105}, \quad x(0) = 0)$

$\dfrac{dx}{dt} = 12 - \dfrac{2x}{105}, \quad x(0) = 0$

b). x_h: $x' = \dfrac{-2x}{105}$

$\dfrac{1}{x} dx = \dfrac{-2}{105} dt$

$\int \frac{1}{x} dx = \int \frac{-2}{105} dt$

$\ln|x| = \dfrac{-2t}{105} + c$

$x = Ae^{\frac{-2}{105}t}$, A arbitrary constant.

Let $A = 1$, $x_h = e^{\frac{-2}{105}t}$, $x = vx_h = ve^{\frac{-2}{105}t}$, sub to $\cancel{x}$

$(ve^{\frac{-2}{105}t})' = 12 - \dfrac{2ve^{\frac{-2}{105}t}}{105}$

$v'e^{\frac{-2}{105}t} = 12$

$v' = 12e^{\frac{2}{105}t}$

$v = \frac{105}{2}(12)e^{\frac{2}{105}t} + c$

$= 630e^{\frac{2t}{105}} + c$

$x = 630e^{\frac{2t}{105}}e^{\frac{-2}{105}t} + ce^{\frac{-2}{105}t}$

$= 630 + ce^{\frac{-2}{105}t}, \quad x(0) = 0)$

$0 = 630 + c$

$c = -630$

$\therefore x(t) = 630 - 630e^{\frac{-2}{105}t}$

$x(t) = 630 - 630e^{\frac{-2}{105}t}$

3. (15 points) Consider the differential equation

$$(t-1)\frac{dx}{dt} + x = t^{1/3}$$

(a) Use the variation of parameters method to solve the equation with the initial value condition $x(0) = 0$.

(b) What is the largest rectangle R in the xt-plane containing the point $(0, 0)$ to which we can apply the existence and uniqueness Theorem? Justify your answer.

a) $(t-1)\frac{dx}{dt} + x = t^{\frac{1}{3}}$

$* \therefore x' = \frac{-x}{t-1} + \frac{t^{\frac{1}{3}}}{t-1}$

x_h: $x' = \frac{-x}{t-1}$

$\frac{1}{x}dx = \frac{-1}{t-1}dt$

$\int \frac{1}{x}dx = \int \frac{-1}{t-1}dt$

$\ln|x| = -\ln|t-1| + C$

$x = A(t-1)^{-1}$, A arbitrary constant

Let $A=0$, $x_h = (t-1)^{-1}$, $x = vx_h = v(t-1)^{-1}$,

sub to $*$

$(v(t-1)^{-1})' = \frac{-v(t-1)^{-1}}{(t-1)} + \frac{t^{\frac{1}{3}}}{t-1}$

$v'(t-1)^{-1} = \frac{t^{\frac{1}{3}}}{t-1}$

$v' = t^{\frac{1}{3}}$

$v = \frac{3}{4}t^{\frac{4}{3}} + c$

$x = \frac{3}{4}t^{\frac{4}{3}}(t-1)^{-1} + \frac{c}{t-1}$

$= \frac{3t^{\frac{4}{3}}}{4(t-1)} + \frac{c}{t-1}$, $x(0)=0$

$0 = \frac{c}{-1}$

$c = 0$.

$\therefore x(t) = \frac{3t^{\frac{4}{3}}}{4(t-1)}$

$x(t) = \cancel{\frac{3t}{t}}\ \frac{3t^{\frac{4}{3}}}{4(t-1)}$

11/11

b) $x' = f(x,t) = \frac{-x}{t-1} + \frac{t^{\frac{1}{3}}}{t-1}$

$f' = \frac{-1}{t-1}$

both $f(x,t)$ and $f'(x,t)$ are continuous everywhere except $t=0$

$t=1$?

$\therefore$ by the uniqueness theorem,

the largest rectangle R:

$R = (-\infty, 1) \times (-\infty, \infty)$

$R: (-\infty, 1) \times (-\infty, \infty)$

4/4

15/15

4. (15 points) Consider the differential equation

10

$$\frac{dx}{dt} = (x+3)^2(1-x)$$

(a) Identify the equilibrium points and draw the phase line diagram of the equation.

(b) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points.

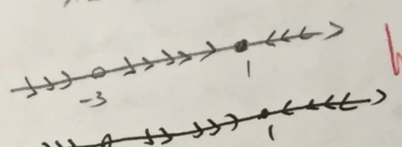
(c) Let $x(t)$ be a particular solution to the equation which satisfies $\cancel{x(1)}$ $x(0) = -4$. Is it possible that $\cancel{x(0)}$ $x(1) = -2$? Justify your answer.

S

a) equilibrium points:

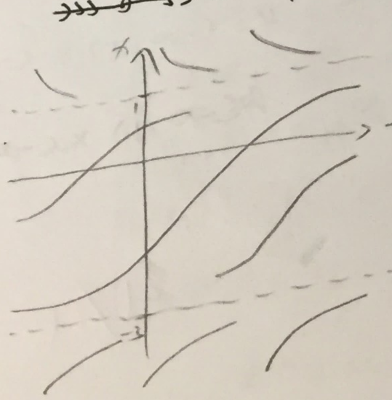
$0 = (x+3)^2(1-x)$

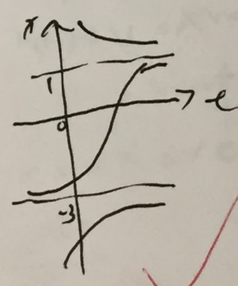
$x = -3$ or 1 $\qquad x = -3$ or 1



how? -1

b)





$x=1$ is stable equilibrium.

~~$x \geq 1$~~ is stable

$x=-3$ is unstable equilibrium

~~$x \geq 3$~~ is unstable

c) By the tx-plane sketch,

for $x < -3$,

$\lim_{t\to\infty} x(t) = -3$.

$\therefore$ for $x(0) = -4 < -3$,

as t grows,

$x(t) < -3$

$\therefore$ $x(1) = -2 > -3$ is not

possible

Not possible

use uniqueness thm

-4