Part I: Multiple choice. Please write your answers (A, B, C, ...) in the boxes on the right.

1. (3 points) Determine values of the constants a and b that make the differential equation exact.

$$(3x^2y - bx^5y^3)\,dx + (ax^3 + x^6y^2)\,dy = 0 \tag{1}$$

A. $a = 2,\ b = -1$

B. $a = 1,\ b = -2$

C. $a = 2, b = 1$

D. $a = -1,\ b = 2$

E. None of the above.

$\frac{\partial P}{\partial y} = 3x^2 - 3bx^5y^2$

$\frac{\partial Q}{\partial x} = 3ax^2 + 6x^5y^2$

$a = 1 \quad b = -2$

B

2. (3 points) Consider the initial value problem

$$y' = (y^2 - 1)\sin^2(ty), \qquad y(0) = 2.$$

Which of the following is true?

$y(0) = 1$

A. $-1 < y(t) < 1$ for all t for which y is defined. $y(0) = -1$

B. $y(t) < \sin^2(t)$ for all t for which y is defined.

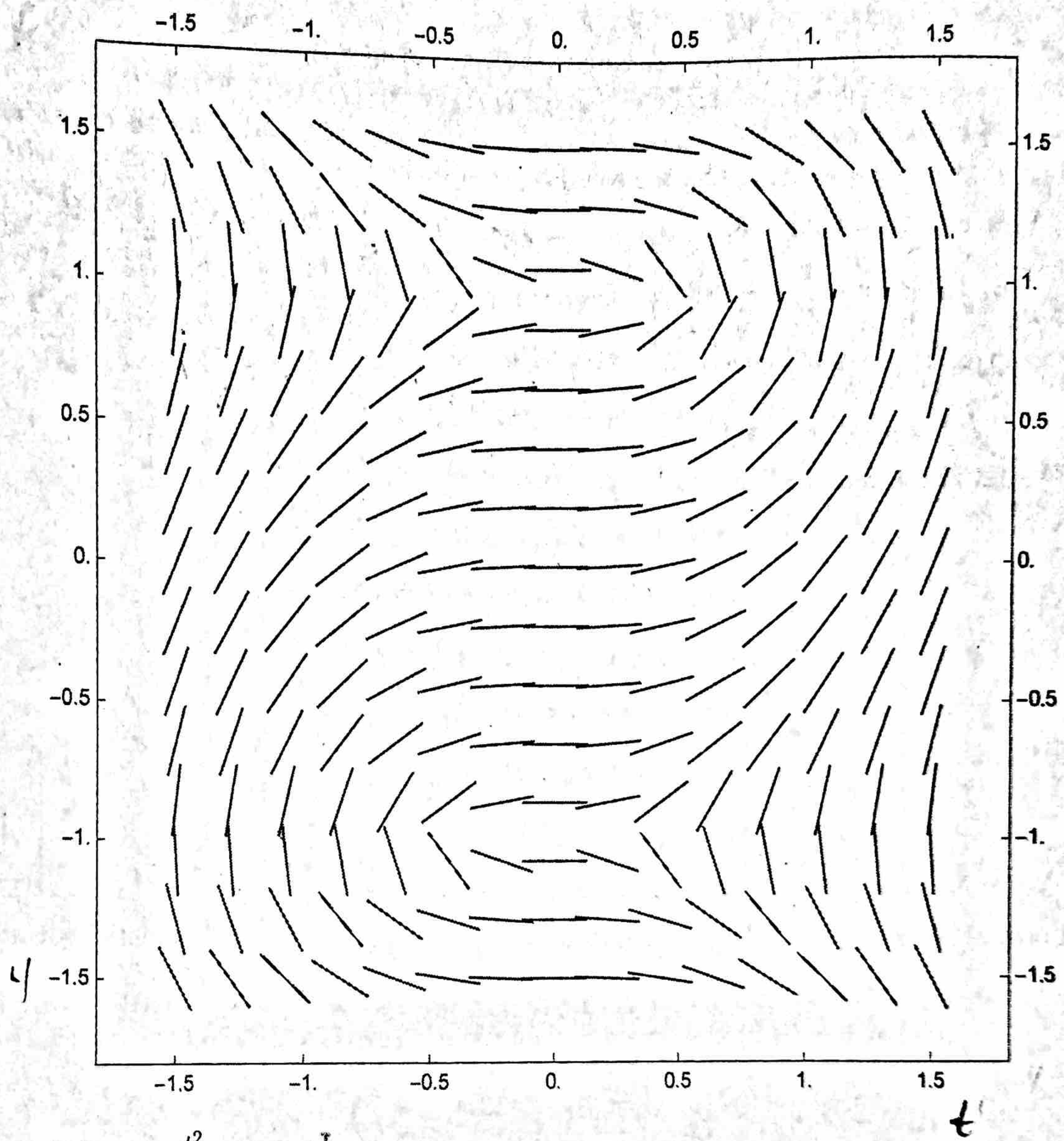
C. $y(t) > 1$ for all t for which y is defined.

D. $y(t) > t^2 - 1$ for all t for which y is defined.

E. None of the above.

C

3. (4 points) The slope field below corresponds to which differential equation?



A. $y' = \dfrac{t^2}{1-y^2}$ $= \dfrac{1^2}{1-(1.5)^2}$ $t=1$

$y=0$

$\dfrac{1}{1-(-1)} = \dfrac{1}{2}$

B. $y' = t^2$

C. $y' = 4y$

D. $y' = ty$

$t=-1$

E. $y' = \dfrac{y^2}{2t+1}$

$y=-1$

$\begin{array}{r} 1.5 \\ \times 1.5 \\ \hline 75 \\ +150 \\ \hline 2.25 \end{array}$

$1-2.25$

-1.25

$\boxed{A}$

4. (4 points) Determine if the equation

$$\left(\frac{y}{x}+8x\right)dx+(\ln x-3)\,dy=0 \tag{2}$$

is exact (you may assume that we are working in a rectangle R in the plane such that $x>0$ for all (x,y) in R). If it is exact, find the solution.

A. $F(x,y)=y\ln x+4x^2-3y$

B. $F(x,y)=-\frac{y}{x^2}+\frac{1}{x}+8$

C. $-\frac{y}{x^2}+\frac{1}{x}+8=C$

D. $y\ln x+4x^2-3y=C$

E. The equation is not exact.

$\frac{\partial P}{\partial y}=\frac{1}{x}$ $\quad\frac{\partial Q}{\partial x}=\frac{1}{x}$

$F(x,y)=\int(\ln x-3)\,dy$

$=y\ln x-3y+g(x)$

$\frac{y}{x}+g'(x)=\frac{y}{x}+8x$

$g'(x)=8x$

$g(x)=4x^2+C$

$F(x,y)=y\ln x-3y+4x^2+C$

$\boxed{D}$

5. (4 points) Which of the following integrating factors is suitable for the differential equation

$(x^2e^x+2xe^x)\sin y\,dx+x^2e^x\cos y\,dy$

$$(x+2)\sin y\,dx+x\cos y\,dy=0? \tag{3}$$

A. $e^{\cos x}$

B. $\sin x$

C. xe^x

D. $1+\frac{1}{x}$

E. None of the above.

$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

$\frac{1}{x\cos y}\left((x+2)\cos y-\cos y\right)$

$\frac{(x+2)-1}{x}=\frac{x+1}{x}=1+\frac{1}{x}$

$e^{\int 1+\frac{1}{x}\,dx}$

$=e^{x+\ln x}$

xe^x

$\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$

$\frac{1}{(x+2)\sin y}\left(\cos y-(x+2)\cos y\right)=\frac{\cos y\,(1-(x+2))}{(x+2)(\sin y)}$

$\frac{\partial P}{\partial y}=(x^2e^x+2xe^x)\cos y$

$\frac{\partial Q}{\partial x}=(x^2e^x+2xe^x)\cos y$

exact

$\boxed{C}$

6. (4 points) The function $\mu(x,y) = \dfrac{1}{x^2+y^2}$ is an integrating factor for the equation

$$(x^2+y^2-x)dx - y\,dy = 0. \tag{4}$$

Use this to solve the differential equation. You may assume that we are working in a rectangle R which does not contain the point $(0,0)$.

- ~~A.~~ $F(x,y) = x - \arctan(x^2+y^2)$
- ~~B.~~ $F(x,y) = x - \frac{1}{2}\ln(x^2+y^2)$
- C. $x - \frac{1}{2}\ln(x^2+y^2) = C$
- D. $x - \arctan(x^2+y^2) = C$
- E. None of the above

$$\frac{x^2+y^2-x}{x^2+y^2}\,dx - \frac{y}{x^2+y^2}\,dy = 0$$

$$F(x,y) = \int -\frac{y}{x^2+y^2}\,dy \qquad u = x^2+y^2 \qquad du = 2y\,dy$$

$$= -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln(x^2+y^2)$$

$\boxed{C}$

7. (3 points) True or false: there exists a differential equation of the form $y' = f(t,y)$ such that f has continuous partial derivatives on a rectangle R containing $(0,0)$ and such that

$$y_1 = 2t \quad \text{and} \quad y_2 = 3t \tag{5}$$

are both solutions in R.

- A. False. The existence theorem forbids it.
- B. False. The uniqueness theorem forbids it.
- C. True. The existence theorem guarantees it.
- D. True. The uniqueness theorem guarantees it.

$\boxed{B}$

Part II: Free Response. Write up a full solution for each problem. A correct answer with an incomplete or incorrect solution will not receive full credit.

8. (6 points) Find the general solution of the linear equation

$$y' - 2y = 4t^3e^{2t}.$$

Box your answer

$y' = 4t^3e^{2t} + 2y$

hg eq: $y' = 2y$

$\int \frac{dy}{y} = \int 2\,dt \Rightarrow \ln|y| = 2t + C$ $\quad A = \pm e^C$

$y = Ae^{2t}$

$y = v(t)e^{2t}$

$y' = v'(t)e^{2t} + \cancel{2v(t)e^{2t}} = 4t^3e^{2t} + \cancel{2v(t)e^{2t}}$

$v'(t)e^{2t} = 4t^3e^{2t}$

$v'(t) = 4t^3$

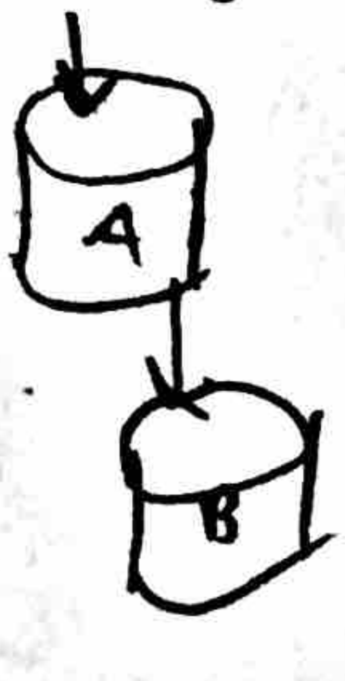
$v(t) = t^4 + C$

$y = (t^4 + C)(e^{2t})$

$$\boxed{y = t^4e^{2t} + Ce^{2t}}$$

9. (8 points) Consider two tanks, labeled tank A and tank B. Tank A contains 150 gal of solution in which is dissolved 12 lbs of salt. Tank B contains 300 gal of solution in which is dissolved 28 lbs of salt. Pure water flows into tank A at a rate of 4 gal/s. There is a drain at the bottom of tank A. The solution leaves tank A via this drain at a rate of 4 gal/s and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 1 gal/s. **Set up, but do not evalute,** a system of differential equations involving the variables x (amount of salt in tank A), y (amount of salt in tank B), and t (time in seconds). Do not forget to include any initial conditions. Box your entire answer

6



A: $\frac{12 \text{ lbs}}{150 \text{ gal}}$ rate in = $\frac{0 \text{ lbs}}{s}$

rate out = $\frac{x(t) \text{ lbs}}{150 \text{ gal}} \cdot \frac{4 \text{ gal}}{s} = \frac{4x(t) \text{ lbs}}{150 \text{ s}}$

$$\frac{dx}{dt} = \frac{-4x(t)}{150 \text{ s}}$$

B: $\frac{28 \text{ lbs}}{300 \text{ gal}}$ rate in = $\frac{4x(t)}{150 \text{ s}}$

rate out = $\frac{y(t) \text{ lbs}}{300 \text{ gal}} \cdot \frac{1 \text{ gal}}{s} = \frac{y(t) \text{ lbs}}{300 \text{ s}}$

$$\frac{dy}{dt} = \frac{4x(t)}{150} - \frac{y(t)}{300} = \frac{8x - 4y}{300}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \frac{-4x}{150}, \quad x(0) = 12 \\ \frac{dy}{dt} &= \cancel{\frac{8x - 4y}{300}}, \quad y(0) = 28 \end{aligned}}$$

10. (5 points) The differential equation

$$y'' + 2t(y')^2 = 0$$

is an example of a *second order differential equation*. The change of variable $v = y'$ (so that $v' = y''$) turns this equation into a first-order separable equation. Using this change of variable, find the particular solution which satisfies

$$y(1) = \frac{\pi}{2}, \qquad y'(0) = 1.$$

Box your answer

$v' + 2t(v)^2 = 0$

$v' = -2t(v)^2$

$\int \frac{dv}{v^2} = \int -2t\,dt \Rightarrow \frac{-1}{v} = -t^2 + C$

$V = \frac{1}{t^2+C}$

$y' = \frac{1}{t^2+C} \Rightarrow 1 = \frac{1}{C} \Rightarrow C = 1$

$y' = \frac{1}{t^2+1}$

$\int 1\,dy = \int \frac{1}{t^2+1}\,dt$

$Y = \arctan(t) + C \qquad Y(1) = \pi/2$

$\frac{\pi}{2} = \cancel{\pi/4} + C$

$C = \pi$

$\boxed{y = \arctan(t) + \cancel{\pi}}$

4

11. (6 points) The differential equation

$$y^2 y' = -y^3 + 3e^{-t} \tag{6}$$

is an example of a Bernoulli equation with degree $n = -2$. $\quad z' = 3y^2 y'$

(a) The change of variables $z = y^3$ turns this equation into a first-order linear differential equation. Using this, write the linear equation above in the form $z' = a(t)z + f(t)$. (I will not be grading your work for this problem, just your answer). $\boxed{\text{Box your answer.}}$

$$z = 3e^{-t} - y^2 y'$$

$$z = 3e^{-t} - \frac{z'}{3}$$

$$\frac{z'}{3} = 3e^{-t} - z$$

$$\boxed{z' = -3z + 9e^{-t}}$$

(b) Solve the differential equation. Your answer should be in the form $\boxed{y(t) = \cdots}$

$$\frac{dz}{z} = -3\,dt \qquad A = \pm e^{c}$$

$$\ln|z| = -3t + c$$

$$z = Ae^{-3t}$$

2

$$z = v(t)e^{-3t}$$

$$z' = v'(t)e^{-3t} + v(t)(-3)e^{-3t} = -3\cancel{(v(t))e^{-3t}} + 9e^{-t}$$

$$v'(t)e^{-3t} = 9e^{-3t}$$

$$v'(t) = 9$$

$$v(t) = 9t + c$$

$$z = (9t + c)(e^{-3t}) \Rightarrow y^3 = 9te^{-3t} + Ce^{-3t}$$

$$\Rightarrow \boxed{y(t) = \sqrt[3]{\cancel{9t}e^{-3t} + Ce^{-3t}}} \quad *$$

$\frac{9}{2}$