Math 33B - Lecture 1
Spring 2018

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your name, section and UID in the space below. For identification purposes, please sign below.

Full Name: [redacted]
Student ID
Section:

Signature:

Question	Points	Score
1	10	10
2	18	17
3	12	10
4	10	8
Total:	50	45

Problem 1.

For each of the following differential equations, determine if it is exact. If the differential equation is exact, solve it.

(a) [4pts.] $\cos x \, dx + \left(1 + \frac{2}{y}\right) \sin x \, dy = 0.$

Exact if $P_y = Q_x$; $\rightarrow P_y = 0$ $\quad Q_x = (1 + \frac{2}{y}) \cos x$ $\rightarrow P_y \neq Q_x \rightarrow \boxed{\text{not exact}}$

(b) [6pts.] $(y \ln y) \, dx + \left(x + \overset{y^{-1}}{\frac{1}{y}} + x \ln y\right) dy = 0.$

$P_y = y \frac{1}{y} + \ln y = 1 + \ln y$, $Q_x = 1 + \ln y$. $P_y = Q_x \rightarrow \boxed{\text{exact}}$.

$F(x,y) = \int y \ln y \, dx + \phi(y) = xy \ln y + \phi(y) \rightarrow \frac{\partial F}{\partial y} = x(y \cdot \frac{1}{y} + \ln y) + \phi'(y) = x + x \ln y + \phi'(y)$

$\cancel{x} + \cancel{x \ln y} + \phi'(y) = Q(x,y) = \cancel{x} + \frac{1}{y} + x \cancel{\ln y} \rightarrow \int \phi'(y) = \int \frac{1}{y} \rightarrow \phi(y) = \ln|y|$

$\rightarrow \boxed{F(x,y) = xy \ln y + \ln|y| = C}$

Problem 2.

Consider the differential equation

$$y' = \frac{dy}{dx} = (y^2 - 1)\sin y.$$

(a) [3pts.] What is the largest rectangle R on the xy-plane on which you can apply the existence theorem? How about the uniqueness theorem? Make sure to justify your answer.

$f(x,y) = (y^2-1)\sin y$ $\quad \frac{\partial f}{\partial y} = (y^2-1)\cos y + (2y)\sin y$. Both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on $(-\infty,\infty)\times(-\infty,\infty) = R$, so $\boxed{R = (-\infty,\infty)\times(-\infty,\infty)}$ is the largest rectangle for both existence and uniqueness theorem. There is no value to make $f(x,y)$ or $\frac{\partial f}{\partial y}$ not continuous.

(b) [3pts.] Identify the equilibrium solutions.

$y' = (y+1)(y-1)\sin y \rightarrow y' = 0$ at $\boxed{y=-1}$, $\boxed{y=1}$ and $\boxed{y = \pi n, \text{ where } n = \text{any integer} \in \mathbb{Z}}$

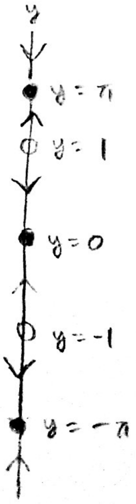
$\sin(y) = 0$

~1.57

~3.14 $\quad$ 0

~4.7

(c) [5pts.] Draw the phase line and identify the stable and unstable points.



$f(-\frac{3\pi}{2}) = (\frac{9\pi^2}{4}-1)(1) = \text{positive}$

$f(-2) = (4-1)\sin(-2) = (\text{pos})(\text{neg}) = \text{neg}$

$f(-\frac{1}{2}) = (\frac{1}{4}-1)(\text{neg}) = \text{pos}$

$f(\frac{1}{2}) = (\frac{1}{4}-1)(\text{pos}) = \text{neg}$

$f(\pi/2) = (\text{pos})(\text{pos}) = \text{pos}$

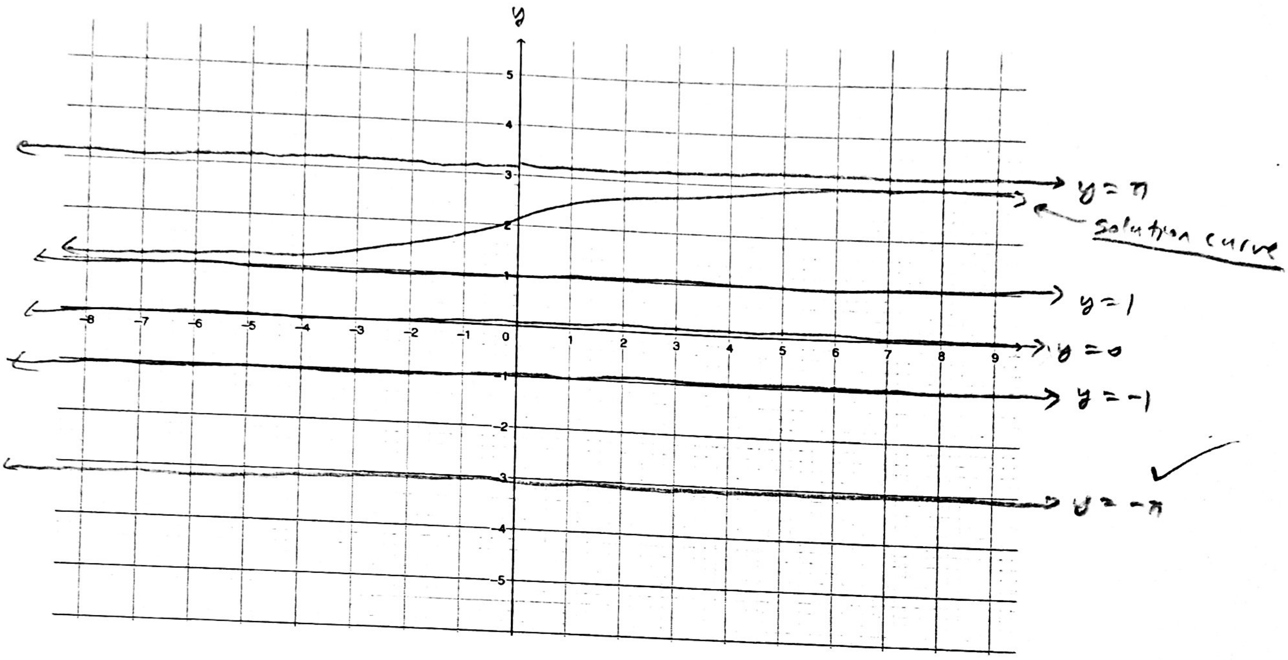
$f(\frac{3\pi}{2}) = (\text{pos})(\text{neg}) = \text{neg}$

Stable: $y = \pi n$, $n = $ odd integer

unstable: $y = -1$, $y = 1$ $\quad -1$

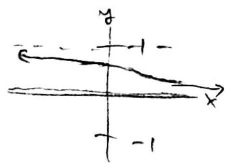
This problem continues from the previous page. Recall that you are studying the differential equation $\frac{dy}{dx} = (y^2 - 1)\sin y$.

(d) [4pts.] Sketch the equilibrium solutions in the xy-plane. These solutions divide the plane in a certain number of regions. Choose one of these regions, and sketch a solution curve inside it.



(e) [3pts.] Let $y(x)$ be the particular solution with initial condition $y(0) = 0.5$. What is the behavior of $y(x)$ as $x \to \infty$?

We know from phase line that for $0 < y < 1$, y is decreasing. But, if $y(0) = 0.5$, the solution curve cannot cross $y = 0$.



So $\boxed{y(x) \to 0}$ as $x \to \infty$

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Problem 3.

Solve each of the following initial value problems. Make sure to discuss the interval of existence.

(a) [6pts.] $\sin x\,dx + y\,dy = 0$, $y(0) = 1$.

$P_y = 0 \quad Q_x = 0 \longrightarrow f(x,y) = \int \sin x\,dx + \phi(y) = -\cos x + \phi(y).$

$\frac{\partial f}{\partial y} = 0 + \phi'(y) = Q(x,y) = y \rightarrow \int \phi'(y) = \int y \rightarrow \phi(y) = \frac{1}{2}y^2$

$F(x,y) = -\cos(x) + \frac{1}{2}y^2 = C \longrightarrow F(0,1) = -\cos(0) + \frac{1}{2}(1) = C \rightarrow -1 + \frac{1}{2} = C \rightarrow C = -\frac{1}{2}$

$\rightarrow \boxed{F(x,y) = -\cos(x) + \frac{1}{2}y^2 = -\frac{1}{2}}$

interval of exist: $\cancel{(-\infty, \infty)}$ -1

Justify -1

4

(b) [6pts.] $xy' + y = e^x$, $y(1) = 2$.

$y' + \frac{1}{x}y = \frac{e^x}{x} \rightarrow u(x) = e^{-\int -\frac{1}{x}dx} = e^{\ln|x|} = x$

$(uy)' = (xy)' = x(y' + \frac{1}{x}y) = x(\frac{e^x}{x}) \longrightarrow xy = \int e^x dx = e^x + C \rightarrow y = \frac{e^x + C}{x}$

$y(1) = 2 = \frac{e^1 + C}{1} \rightarrow C = 2 - e \rightarrow \boxed{y = \frac{e^x + 2 - e}{x}}$ ✓

, $x \neq 0$

interval of exist: $(0, \infty)$

ok.

Justify better next time

not $(-\infty, 0)$ b/c $y(1) = 2$

6.

$\frac{d}{dx}\left(\ln|\sec(\pi x)+\tan(\pi x)|\right) = \frac{1}{\quad}\left(\sec(\pi x)\tan(\pi x)(\pi) + \sec^2(\pi x)(\pi)\right)$

$\frac{d}{dx}\left(\ln|\sec x + \tan x|\right) = \frac{1}{\sec x + \tan x}\left(\sec\tan + \sec^2\right) = \frac{\sec(\tan + \sec)}{\sec + \tan} = \sec(x)$ ✓

Problem 4.

Consider the differential equation $\frac{\mathrm{dy}}{\mathrm{dx}} = \sin x \cos(\pi y)$. [This problem continues onto the next page]

(a) [4pts.] Find the general solution of this differential equation. You can leave your solution in an implicit form, if you want.

$\int \frac{1}{\cos(\pi y)}\,dy = \int \sin x\,dx \rightarrow \int \sec(\pi y)\,dy = -\cos x + C_1 \rightarrow \frac{1}{\pi}\ln|\sec(\pi y) + \tan(\pi y)| = -\cos x + C$

+4 Thorough enough

$\ln|\sec(\pi y) + \tan(\pi y)| = -\pi\cos x + \pi C \rightarrow \sec(\pi y) + \tan(\pi y) = e^{-\pi\cos x + \pi C} = De^{-\pi\cos x}$

$\frac{1}{\cos(\pi y)} + \frac{\sin(\pi y)}{\cos(\pi y)} = \frac{\sin(\pi y)+1}{\cos(\pi y)} = De^{-\pi\cos x} \rightarrow \sin(\pi y) + 1 = \cos(\pi y)De^{-\pi\cos x}$

$\sin(\pi y) - \cos(\pi y)De^{-\pi\cos x} = -1$

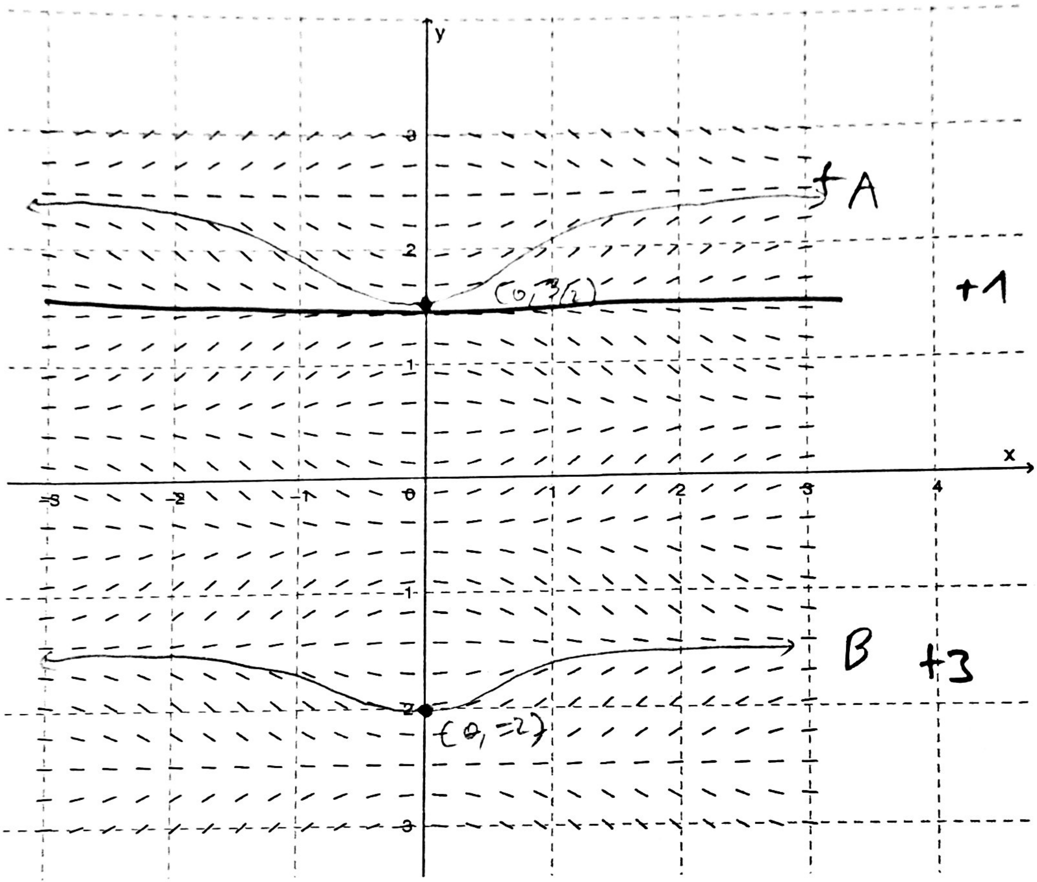
$\int \sec(\pi y)\,dy$

$u = \sec(\pi y) \quad dv = dy$

$du = \sec(\pi y)\tan(\pi y) \quad v = y$

Recall that we are studying the differential equation $\frac{dy}{dx} = \sin x \cos(\pi y)$.

(b) [6pts.] This is the direction field of the above differential equation.



Sketch the solution curves for the initial value problems corresponding to the two follow initial conditions:

(A) $y(0) = \frac{3}{2}$.

(B) $y(0) = -2$.

Make sure to label your solution curves as (A) and (B), correspondingly to each initial condition.