P

MIDTERM 2 B

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Math33B
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section: 2B

Problem	Points	Score
1	8	8
2	9	5
3	4	4
4	8	7
5	8	8
6	3	3
Total	40	~~[illegible]~~ 35

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) **Use a PEN** to record your final answers.
(3) If you need **more space**, use the extra page at the end of the exam.
(4) NO Calculators, computers, books or notes of any kind are allowed.
(5) Show your work. Unsupported answers will not receive full credit.
(6) Good Luck!

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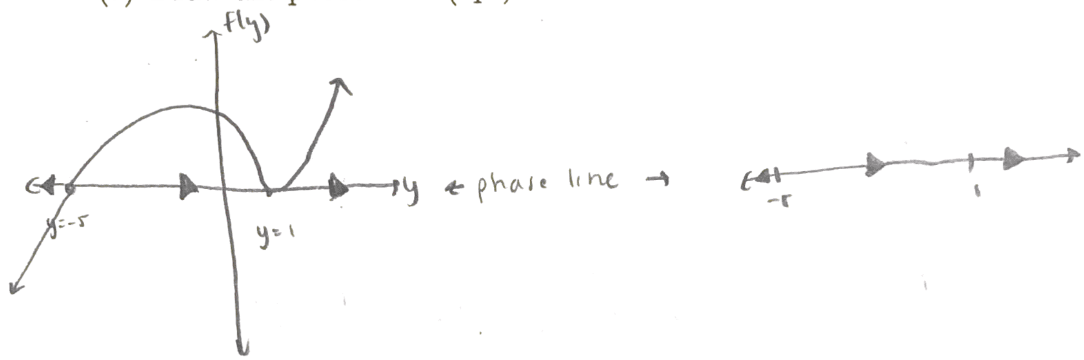
Exercise 1. (8pt)
Consider the autonomous first-order differential equation.

$$y' = (y-1)^2(y+5)$$

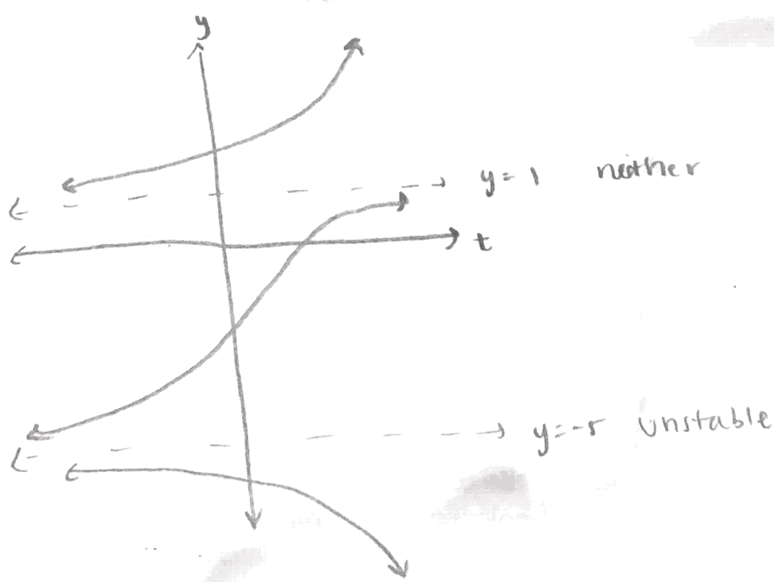
(1) Draw the phase line. (3pt)



(2) What are the equilibrium solutions, which are stable, and which are unstable? (2pt)

stable: none
unstable: $y=-5$
necther: $y=1$

(3) Sketch the graph of at least one solutions between each pair of adjacent equilibrium solutions. (3pt)



Exercise 2. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{(x+1)e^t}{x-2}$$

$\frac{x-2}{x+1}dx = e^t dt$

(1) Can you apply the existence and uniqueness theorem to the initial value problem $x_0(0) = 1$? Justify your answer and give the biggest rectangle in which you can apply it containing the given point (if it exists). (5 pt)

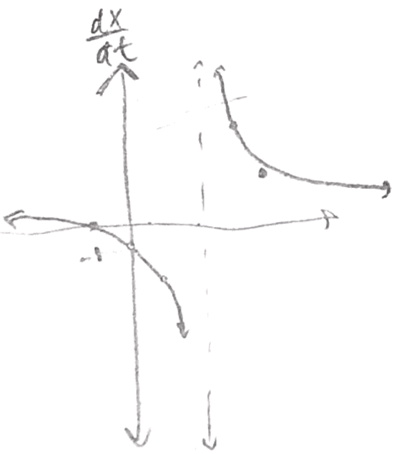
$\frac{dx}{dt} = \frac{(x+1)e^t}{x-2}$ $\quad x \neq 2 \quad t = \forall \mathbb{R}$

$\frac{d\left(\frac{(x+1)e^t}{x-2}\right)}{dx} = e^t\left(\frac{x-2-x-1}{(x-2)^2}\right) = e^t \frac{-3}{(x-2)^2}$ $\quad x \neq 2 \quad t = \forall \mathbb{R}$

if $x(0) = 1$ $\quad 1 < 2$

so x range $(-\infty, 2)$

t has no restrictions, has to contain 0



$\boxed{\text{rectangle } \underset{t}{(-\infty, \infty)} \times \underset{x}{(-\infty, 2)}}$

5

t α

(2) Can $x_0(2) = -2$ ($x_0(t)$ is the solution to the initial value problem in part 1))? Justify your answer. (4pt)

~~yes~~ It also exists in the rectangle $(-\infty, \infty) \times (-\infty, 2)$

this please No which means ~~that~~ there is a unique solution

$X_0(2) = -2$ is unique from $X_0(0) = 1$

In addition, the graph of $\frac{dx}{dt}$ is negative when

$X = 1$ as in the initial condition so it would make

sense that it gets more negative as t increase

b/c e^t always positive.

~~e^t is always positive so it doesn't make sense~~

~~that X would become negative as t increases~~

O

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Exercise 3. (4pt) Find a **particular** solution to the following two differential equations

$$y'' - 2y' - 3y = -4e^{3t}$$

$y_p(t)$ of the form ae^{3t}

$y = ae^{3t}$ $y' = 3ae^{3t}$ $y'' = 9ae^{3t}$

$9ae^{3t} - 2(3ae^{3t}) - 3ae^{3t} = 0$ nope

2nd try

$y_p(t)$ of the form ate^{3t}

$y = ate^{3t}$ $y' = at3e^{3t} + ae^{3t}$ $y'' = a9te^{3t} + 3ae^{3t} + 3ae^{3t}$

$6ae^{3t} + 9ate^{3t} - 2(3ate^{3t} + ae^{3t}) - 3(ate^{3t}) = -4e^{3t}$

$6ae^{3t} + \cancel{9ate^{3t}} - \cancel{6ate^{3t}} - 2ae^{3t} - \cancel{3ate^{3t}}$

$4ae^{3t} = -4e^{3t}$

$a = -1$

$$\boxed{y_p = -te^{3t}}$$

Check:

$y_p = -te^{3t}$ $y' = -3te^{3t} - e^{3t}$ $y'' = -9te^{3t} - 3e^{3t} - 3e^{3t}$

$\cancel{-9te^{3t}} - 6e^{3t} + \cancel{6te^{3t}} + 2e^{3t} + \cancel{3te^{3t}}$

$-6e^{3t} + 2e^{3t} = -4e^{3t}$

✓

Exercise 4. (8pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$A = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$ $\quad (-1-\lambda)(-\lambda) - 6 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda+3)(\lambda-2)$

$\lambda = -3, 2$

$A+3I \quad \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

$A-2I \quad \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$

eigenvectors

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

general solution: $y(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ✓

$y(0) = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ split up equations

$0 = C_1 e^{-3(0)} + 2C_2 e^{2(0)}$ $\qquad -5 = -C_1 e^{-3(0)} + 3C_2 e^{2(0)}$

$0 = C_1 + 2C_2$ $\qquad -5 = -C_1 + 3C_2$

$-5 = -C_1 + 3C_2$

$+$

$-5 = 5C_2$

~~$C_2 = -1$ ✓ $C_1 = 1$~~

$C_2 = -1$

$C_1 = 2$

-1pt

$$\boxed{y(t) = e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

(✓)

Exercise 5. (8pt) Consider the differential equation

$$y'' + \frac{1+x}{x}y' - \frac{1}{x}y = 0$$

(1) check that $1+x$ and $\frac{2x^2+6x+4}{x+2}$ are solutions to the above equation. (4pt)

$y = 1+x \quad y' = 1 \quad y'' = 0$

plug in y'', y' 랑 y

$0 + \frac{1+x}{x}(1) - \frac{1}{x}(1+x) = 0$

$\frac{1+x}{x} - \frac{1+x}{x} = 0$ yes ∴ a solution

$y = \frac{2x^2+6x+4}{x+2} = \frac{2(x^2+3x+2)}{x+2} = \frac{2\cancel{(x+2)}(x+1)}{\cancel{x+2}} = 2x+2$

$y' = 2 \quad y'' = 0$

plugin y'' y' 랑 y

$0 + \frac{1+x}{x}(2) - \frac{1}{x}(2x+2) = 0$

$\frac{2+2x}{x} - \frac{2+2x}{x} = 0$ yes ∴ a solution

(2) Do they form a fundamental set of solutions? Justify your answer. (4pt)

$y_1 = 1+x \quad y_2 = 2+2x$

$\frac{y_1}{y_2} = \frac{1}{2}$ which is a constant

therefore these 2 solutions are $\boxed{\text{linearly dependent}}$ on each other and $\boxed{\text{don't form a fundamental set}}$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1+x & 2+2x \\ 1 & 2 \end{vmatrix} \quad 2(1+x) - 2 - 2x = 0$$

wronskian is also 0, showing they are Linearly dependent and thus not a fundamental set

Exercise 6. (3pt)
Consider the differential equations

$$(2yx)dx + (yx^2 + y)dy = 0$$

Find the integrating factor for the above equation. (you do NOT have to solve the equation)
(Hint: it only depends on y)

$2yx\,dx + yx^2 + y\,dy = 0$

$\frac{dP}{dy} = 2x \qquad \frac{dQ}{dx} = 2xy$

$\mu(y) = \frac{1}{P}\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)$

$= \frac{1}{2yx}(2xy - 2x)$

$= \frac{\cancel{2x}(y-1)}{\cancel{2x}y} = \frac{y-1}{y}$

$e^{\int \frac{y-1}{y}} = e^{\int \frac{y}{y} - \int \frac{1}{y}} = e^{y - \ln y} = e^{y} e^{-\ln y} = e^{y}\frac{1}{y} = \frac{e^{y}}{y}$ $\quad \boxed{\mu(y) = \frac{e^{y}}{y}}$ ✓

$2xe^{y}dx + x^2e^{y} + 1\,dy$

$\frac{dP}{dy} = 2xe^{y} \qquad \frac{dQ}{dx} = 2xe^{y}$

integration factor works