MIDTERM 2 A

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Math33B
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section: D

Problem	Points	Score
1	8	8
2	9	8
3	4	4
4	8	8
5	8	8
6	3	2
Total	40	38

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) **Use a PEN** to record your final answers.
(3) If you need **more space**, use the extra page at the end of the exam.
(4) NO Calculators, computers, books or notes of any kind are allowed.
(5) Show your work. Unsupported answers will not receive full credit.
(6) Good Luck!

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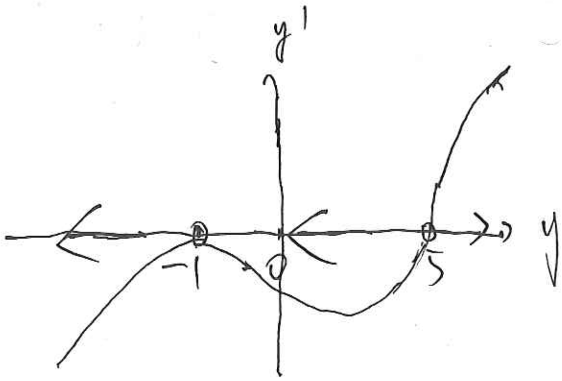
Exercise 1. (8pt)
Consider the autonomous first-order differential equation.

$$y' = (y+1)^2(y-5)$$

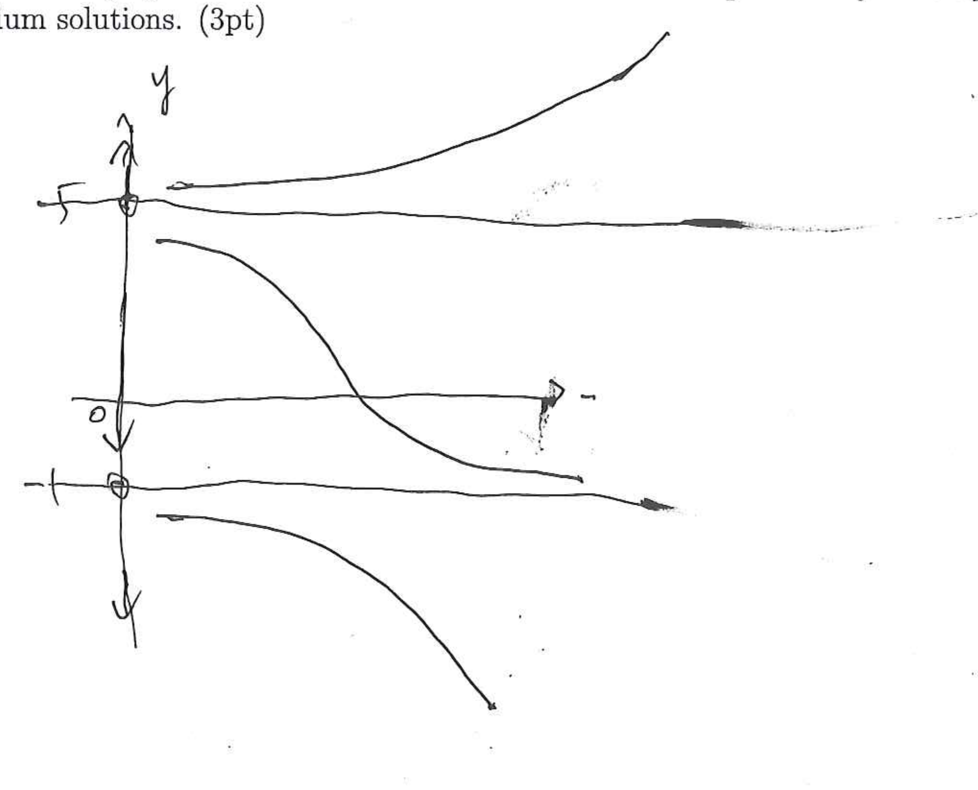
(1) Draw the phase line. (3pt)



(2) What are the equilibrium solutions, which are stable, and which are unstable? (2pt)

$y=5$ and $y=-1$ are equilibrium solutions
neither is stable
$y=5$ is unstable
$y=-1$ is undetermined

(3) Sketch the graph of at least one solutions between each pair of ~~adjacent~~ equilibrium solutions. (3pt)



Exercise 2. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{(x-1)e^t}{x+2} = \frac{e^t x - e^t}{x+2}$$

(1) Can you apply the existence and uniqueness theorem to the initial value problem $x_0(0) = 2$? Justify your answer and give the biggest rectangle in which you can apply it containing the given point (if it exists). (5 pt)

x' is continuous when $x \neq -2$

$$\frac{\partial F}{\partial x} = \frac{e^t(x+2) - (x-1)e^t}{(x+2)^2} = e^t \frac{3}{(x+2)^2}$$

it's continuous when $x \neq -2$

the biggest rectangle containing the initial value is $(-\infty, \infty) \times (-2, \infty)$

$x_0(0) = 2$ is in that rectangle

$\therefore$ ~~it's~~ it exists and it's unique

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(2) Can $x_0(2) = 0$ ($x_0(t)$ is the solution to the initial value problem in part 1))? Justify your answer. (4pt)

$X = 1$ is a solution

if $X_0(0) = 2$, $X_0(2) = 0$

there must be a point of X_0 where

$X_0 = 1$ ~~yes~~ well.. there is a function $x(t) = 1$

since it's unique in that rectangle

it can't be like that

$\therefore$ No!

3/4

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Exercise 3. (4pt) Find a **particular** solution to the following two differential equations

$$y'' - 3y' - 4y = 10e^{4t}$$

$y = Ae^{4t}$ $y' = 4Ae^{4t}$ $y'' = 16Ae^{4t}$

$y'' - 3y' - 4y = (16 - 12 - 4)Ae^{4t} = 10e^{4t}$ $\rightarrow$ not possiblo

$y = Ate^{4t}$ $y' = Ae^{4t} + 4Ate^{4t}$

$y'' = 4Ae^{4t} + 4Ae^{4t} + 16Ate^{4t}$

$\therefore y'' - 3y' - 4y = 8Ae^{4t} + 16Ate^{4t} - 3Ae^{4t} - 12Ate^{4t}$
$- 4Ate^{4t}$

$= 5Ae^{4t}$

$= 10e^{4t}$

$\therefore A = 2$

$\therefore y = 2te^{4t}$

Exercise 4. (8pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

first find the eigenvalue and eigenvector

$(1-\lambda)(0-\lambda)-6=0$

$(\lambda-1)\lambda - 6 = 0$

$\lambda^2-\lambda-6=0$

$(\lambda-3)(\lambda+2)=0$

$\lambda_1=3, \lambda_2=-2$

$\lambda=3$

$A-\lambda I = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$

$\therefore \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda=-2$

$A-\lambda I = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

$\therefore \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\therefore$ the general solution is

$\vec{y} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\because \vec{y}(0) = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$

we plug it in and have

$C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 3 & -5 \end{bmatrix}$ use Gauss elimination

$\Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \quad \therefore C_1=-2, C_2=-1$

$\therefore$ the solution is

$\vec{y} = -2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{*} -1\, e^{-2t} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Exercise 5. (8pt) Consider the differential equation

$$y'' + \frac{1-x}{x}y' + \frac{1}{x}y = 0$$

(1) check that $1 - x$ and $\frac{3x^2-9x+6}{2-x}$ are solutions to the above equation. (4pt)

$(1-x)' = -1$

$(1-x)'' = 0$

$0 - \frac{1-x}{x} + \frac{1-x}{x}$

$= 0$

$\frac{3x^2-9x+6}{2-x}$

$= \frac{3(x^2-3x+2)}{2-x}$

$= \frac{3(x-1)(x-2)}{2-x}$

$= 3(1-x)$

$= 3-3x$

$(3-3x)' = -3$

$(3-3x)'' = 0$

$0 + \frac{1-x}{x}(-3) + \frac{3-3x}{x}$

$= 0$

$\therefore$ both are solutions

(2) Do they form a fundamental set of solutions? Justify your answer. (4pt)

$$W = d\left(\begin{bmatrix} 1-x & 3-3x \\ -1 & -3 \end{bmatrix}\right)$$

$$= 3x-3-(3x-3) = 0$$

$\therefore$ there are linearly dependent

$\therefore$ they don't form a fundamental set of solutions

Exercise 6. (3pt)
Consider the differential equations

$$(y^2x + x)dx + (3xy)dy = 0$$

Find the integrating factor for the above equation. (you do NOT have to solve the equation)
(Hint: it only depends on x)

$\cancel{h = \frac{1}{3xy}(2xy}$

$\cancel{2y}$ $\frac{\partial P}{\partial y} = 2xy$

$\frac{\partial Q}{\partial x} = 3y$

$\therefore h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= \frac{1}{3xy}(2xy - 3y)$

$= \frac{1}{3x}(2x - 3)$

it only depends on x

$\therefore u = e^{\int h dx}$

$\int \frac{1}{3x}(2x-3)\,dx$

$= \int \frac{2x}{3x}dx - \int \frac{3}{3x}dx$

$\cancel{= \int \frac{2}{3}dx - \int \frac{1}{3x}d3x}$

$\cancel{= \frac{2}{3}x - \ln 3x}$

$\cancel{\therefore u = e^{\frac{2}{3}x - \ln 3x}}$

$\cancel{= e^{\frac{2}{3}x} \cdot (e^{\ln 3x})^{-1}}$

$\cancel{= \frac{e^{\frac{2}{3}x}}{3x}}$

-1pt

$= \frac{2}{3}x - \cancel{3}\int \frac{1}{x}dx$

$= \frac{2}{3}x - 3\ln x$

$\therefore u = e^{\frac{2}{3}x - 3\ln x}$

$= \frac{e^{\frac{2}{3}x}}{e^{3\ln x}}$

$= \frac{e^{\frac{2}{3}x}}{x^3}$ (✓)

Extra page