33B Midterm 2

TOTAL POINTS

39 / 40

QUESTION 1

auto. Equation 11 pts

1.1 Phase Line 3 / 3

✓ **- 0 pts** Correct

- 1 pts arrows missing/wrong

- 1 pts graph wrong somewhere

- 1 pts missing scale

- 3 pts not the phase line

- 1 pts too many zero's in the graph

1.2 Eq. Solutions 3 / 3

✓ **- 0 pts** Correct

- 1 pts -3 no conclusion

- 1 pts 2 is stable

- 1 pts 5 is unstable

- 2 pts Stable/unstable?

1.3 Graph sketch 2 / 2

✓ **- 0 pts** Correct

- 2 pts no solution/wrong solution

- 1 pts graph between -3 and 2 wrong/missing

- 1 pts graph between 2 and 5 wrong/missing

1.4 particular solution 3 / 3

✓ **+ 3 pts** Correct

+ 1 pts No

+ 1 pts Uniqueness theorem can be applied

+ 0 pts wrong/no answer

+ 1 pts cannot cross the equilibrium solution y(t) = 2

QUESTION 2

Existence and Uniqueness 8 pts

2.1 Apply? Rectangle? 4 / 5

✓ **+ 2 pts** continuous

✓ **+ 2 pts** derivative continuous

✓ **+ 1 pts** rectangle

+ 0 pts no points

- 1 Point adjustment

x cannot be between -2 and 2

2.2 x_0(2)=5? 3 / 3

✓ **+ 1 pts** Correct

✓ **+ 2 pts** justification

+ 0 pts no points

QUESTION 3

3 Particular Solution 6 / 6

✓ **- 0 pts** Correct

- 1 pts Mixed up a minus sign

- 3 pts Didn't try the right guess (ae^3t)

- 6 pts Didn't attempt method of undetermined coefficients.

- 1 pts Incorrect arithmetic in finding constant.

- 1 pts Incorrect multiplication

- 1 pts Put constant in solution

- 3 pts Forgot to include an undetermined coefficients in MOC.

QUESTION 4

2. order equation constant coefficients 5 pts

4.1 verify solutions 3 / 3

✓ **- 0 pts** Correct

- 2 pts Didn't explicitly check boundary conditions

- 1 pts Only checked one boundary condition

- 1 pts Didn't correctly check that they satisfy the ODE.

4.2 existence and uniqueness? 2 / 2

✓ **- 0 pts** **Correct**

- 2 pts Didn't understand that solution was non-unique.

- 2 pts Didn't state that the boundary conditions being defined at different times renders the existence and uniqueness theorem irrelevant.

- 1 pts Not clear if you actually meant that the "initial" conditions are defined at different times.

QUESTION 5

2. order equation 7 pts

5.1 verify solutions 4 / 4

✓ **- 0 pts** **Correct**

- 2 pts incorrect calculation

- 4 pts incorrect calculation

5.2 fundamental set 3 / 3

✓ **- 0 pts** **Correct**

- 1 pts conclusion is incorrect,

- 1 pts some work, calculation incorrect,

- 3 pts conclusion incorrect, wrong calculation

- 2 pts some work

QUESTION 6

6 planar system 3 / 3

✓ **- 0 pts** **Correct**

- 2 pts incorrect, but some work

- 1 pts minor mistake

- 3 pts no work

MIDTERM 2

11/16/2018

Math33B
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Name:

UID:

section:

Problem	Points	Score
1	11	
2	8	
3	6	
4	5	
5	7	
6	3	
Total	40	

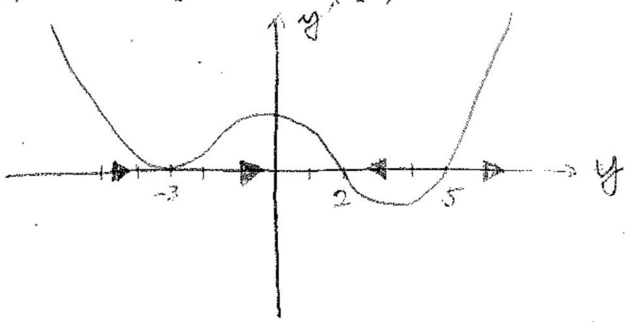
Instructions

(1) This exam has 6 problems. Make sure you have all pages.
(2) Enter your name, SID number, and discussion section on the top of this page.
(3) **Use a PEN** to record your final answers.
(4) If you need **more space**, use the extra page at the end of the exam.
(5) NO Calculators, computers, books or notes of any kind are allowed.
(6) Show your work. Unsupported answers will not receive full credit.
(7) Good Luck!

Exercise 1. (11pt)
Consider the autonomous first-order differential equation.

$$y' = (y+3)^2(y-2)(y-5)$$

(1) Draw the phase line. (3pt)



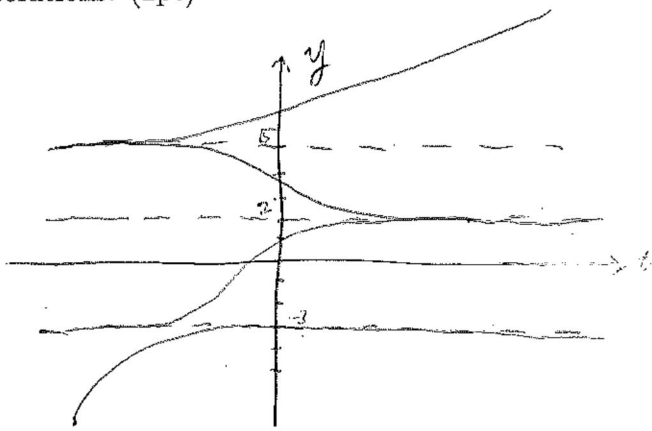
(2) What are the equilibrium solutions? Which are stable, and which are unstable? (3pt)

$y = -3$ no conclusion

$y = 2$ stable

$y = 5$ unstable

(3) Sketch the graph of at least one solutions between each pair of adjacent equilibrium solutions. (2pt)



(4) Let $y_p(t)$ be a particular solution to the equation which satisfies $y_p(0) = 0$. Is it possible that $y_p(2) = 2$? Justify your answer. (3pt)

Let $f(y) = (y+3)^2(y-2)(y-5)\left(= (y^2+6y+9)(y^2-7y+10) = y^4 - y^3 + 6y^2 - 3y + 90\right)$

$f'(y)$ is a polynomial of degree 3. $\left(f'(y) = 4y^3 - 3y^2 + 12y - 3\right)$.

As both $f(y)$ and $f'(y)$ are continuous for any (t, y),

~~exist~~ uniqueness applies to the entire ty-plane.

Given that $y(t) = 2$ is a solution to the equation, y_p

cannot intersect $y(t) = 2$ without violating uniqueness. However

it does intersect at $(2, 2)$. As such such a y_p $\boxed{\text{cannot exist.}}$ $\square$

Exercise 2. (8pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{\sqrt{x^2-4}}{t^2}.$$

(1) Can you apply the existence and uniqueness theorem to the initial value problem $x_0(1) = 6$? Justify your answer and give the biggest rectangle in which you can apply it containing the given point (if it exists). (5 pt)

Let $f(t,x) = \frac{\sqrt{x^2-4}}{t^2}$: discontinuous at $t=0$, $x=-2$, $x=2$.
continuous on $(0,\infty)\times[2,\infty)$

$\frac{\partial f}{\partial x} = \frac{\cancel{2}x}{\cancel{2}t^2\sqrt{x^2-4}} = \frac{x}{t^2\sqrt{x^2-4}}$; discontinuous at $t=0$, $x=-2$, $x=2$
continuous on $(0,\infty)\times(2,\infty)$

Given both $f(1,6)$ and $\left.\frac{\partial f}{\partial x}\right|_{(t,x)=(1,6)}$ are defined, and $f(t,x)$ and $\frac{\partial f}{\partial x}$ are continuous at that point, the existence and uniqueness theorem <u>does</u> apply to that IVP.

Biggest rectangle: $\boxed{(0,\infty)\times(2,\infty)}$

(2) Can $x_0(2) = 5$ ($x_0(t)$ is the solution to the initial value problem in part 1))?(1pt)
Justify your answer. (2pt)

No, it cannot.

By existence theorem, if $x_0(1) = 6$ were to be true, x_0 must be continuous in $(0, \infty) \times (2, \infty)$. At the same time, we see $\frac{dx}{dt} = \frac{\sqrt{x^2-4}}{t^2} > 0$ for all $(t, y) \in (0, \infty) \times (2, \infty)$.

Thus, it is impossible ~~that~~ to have $x_0(2) < x_0(1)$. $\square$

Exercise 3. (6pt) Find a **particular** solution to the following differential equation

$$3y'' + 2y' - y = -4e^{3t}.$$

Guess $y_p(t) = ae^{3t}$.

We have $y_p'(t) = 3ae^{3t}$.

$y_p''(t) = 9ae^{3t}$.

$$a\cancel{e^{3t}}(27 + 6 - 1) = -4\cancel{e^{3t}}$$

$$32a = -4$$

$$a = -\frac{1}{8}.$$

$$\boxed{y_p(t) = -\frac{1}{8}e^{3t}}$$

Check: $y_p'(t) = -\frac{3}{8}e^{3t}$.

$y_p''(t) = -\frac{9}{8}e^{3t}$.

$$e^{3t}\left(-\frac{27}{8} - \frac{6}{8} + \frac{1}{8}\right) = -4e^{3t}. \checkmark$$

Exercise 4. (5pt) Consider the following problem:

$$y'' + y = 0 \qquad y(0) = 0 \;\; y'(\pi/2) = 0$$

(1) Show that $y(t) = C \cdot \sin(t)$ is a solution for any constant C. (3pt)

$y'(t) = C\cos(t).$

$y''(t) = -C\sin(t).$

$y'' + y = -C\sin(t) + C\sin(t) = 0$ ✓.

$y(0) = C\sin(0) = 0.$ ✓

~~y(t)~~ $y'(\frac{\pi}{2}) = C\cos(\frac{\pi}{2}) = 0.$ ✓.

$\underset{= C\sin t}{y(t)}$ satisfies all conditions.

(2) Why does this not violate the 2. order existence and uniqueness theorem? (2pt)

The 2nd order existence and uniqueness theorem stipulates that the initial values for $y(t_0)$ and $y'(t_0)$ are at the same $t = t_0$. However in this IVP, $y(0)$ is given but ~~y~~ ~~y~~ $y'(\frac{\pi}{2})$ is given in lieu of $y'(0)$. As such, the assumptions of the 2nd order existence and uniqueness theorem are not satisfied, and the theorem cannot be applied.

Exercise 5. (8pt) Consider the differential equation

$$y'' + \frac{1+x}{x}y' - \frac{1}{x}y = 0$$

(1) Check that $1 + x$ and $\frac{2x^2+6x+4}{x+2}$ are solutions to the above equation. (4pt)

Check $y(x) = 1+x$.

$y'(x) = 1$

$y''(x) = 0$.

$0 + \frac{1+x}{x} - \frac{1}{x}(1+x) = \frac{1+x}{x} - \frac{1+x}{x} = 0$. ✓.

Check $y(x) = \frac{2x^2+6x+4}{x+2} = \frac{2(x^2+3x+2)}{x+2} = \frac{2(x+2)(x+1)}{x+2} = \begin{cases} \text{undefined}, & \text{if } x = -2 \\ 2(x+1), & \text{otherwise.} \end{cases}$

$y'(x) = 2 \quad (x \neq -2)$

$y''(x) = 0, \quad (x \neq -2)$

$0 + \frac{1+x}{x} \cdot 2 - \frac{2(x+1)}{x} = 0$ ✓.

Both are solutions.

(2) Do they form a fundamental set of solutions?(1pt) Justify your answer. (2pt)

No, they do not.

Let $y_1(x) = 1+x$, $y_1'(x) = 1$.

$y_2(x) = \frac{2x^2+6x+4}{x+2}$, $y_2'(x) = \frac{4x+6}{x+2} - \frac{2x^2+6x+4}{(x+2)^2}$

$= \frac{4x^2+14x+12-2x^2-6x-4}{(x+2)^2}$

$= \frac{2x^2+8x+8}{(x+2)^2}$

$= \frac{2(x+2)}{x+2}$.

$W_{y_1,y_2} = y_1 \cancel{(x)} y_2' - y_2 y_1'$

$= (x+1)\frac{2(x+2)}{x+2} - \frac{2(x+2)(x+1)}{x+2}$

$= 0$.

As y_1 and y_2 are linearly dependent, they do not form a fundamental set of solutions.

Exercise 6. (3pt)
Consider the second order equation

$$y'' - 2e^t y' - \tan(t)y = \sqrt{t^2+1}.$$

Write this equations as a planar system of first-order equations.

Let $v = y'$.

$$\begin{cases} y' = v. \\ v' = 2e^t v + \tan(t)\cdot y + \sqrt{t^2+1} \end{cases}$$