33B midterm 1

TOTAL POINTS

37 / 40

QUESTION 1

integration factor 8 pts

1.1 integration factor 4 / 4

✓ **- 0 pts Correct**

- 1 pts minor mistake

- 4 pts no work

- 3 pts subtle work, try to find h(x) but equation incorrect

- 2 pts get h(x),but not u(x)

- 2 pts get u(x) but without details; know how to get u(x) but calculate incorrectly

1.2 solve 4 / 4

✓ **- 0 pts Correct**

- 1 pts solution should be in form of F(x,y) = c

- 4 pts no work

- 3 pts know need to do partial integration, but incorrect.

- 2 pts correct form F = \phi + xxxx, but \phi incorrect ; or the other way around.

- 1 pts minor mistake

QUESTION 2

separable eon 12 pts

2.1 explicit solution 5 / 5

✓ **+ 1 pts Separating the Equation**

✓ **+ 1 pts Partial Fractions**

✓ **+ 1 pts Computing Integral**

✓ **+ 1 pts Log Rule Application**

✓ **+ 1 pts Computing Solution**

+ 2 pts Bernoulli Transformation

+ 1 pts Integrating Factor

+ 2 pts Rest of Bernoulli Solution

+ 0 pts No points

Technically correct, but the arbitrariness of C means you can drop the absolute value.

2.2 y(1) = 2 2 / 2

✓ **+ 2 pts Correct Answer**

+ 1.5 pts Correct Answer, Wrong Solution

+ 1 pts Knowing the Process

+ 0 pts No points

2.3 interval of existence 1 / 3

+ 1 pts Knowing 0 is not included

+ 1 pts Correct for their function

+ 1 pts Correct

✓ **+ 1 pts Knowing 2 is not included.**

+ 0 pts No points

2.4 y(1) = 0 2 / 2

✓ **+ 2 pts Correct Answer**

+ 1 pts Correct Answer, but on accident

+ 0 pts No points

QUESTION 3

3 mixing problem 6 / 7

- 1 pts Identifying x'= rate in- rate out, rate in = 4

- 2 pts Identify rate out = x/(50+t)

✓ **- 1 pts Find an integrating factor or homogeneous solution**

- 2 pts Find the general solution

- 1 pts Incorporate the initial condition.

- 0 pts Correct

- 1 pts Accidentally made equation Homogeneous/ too simple.

- 1 pts Forgot a factor of 2 in rate out.

QUESTION 4

exact eqn 7 pts

4.1 not exact 3 / 3

✓ **- 0 pts Correct**

- 3 pts No answer

- **2 pts** wrong derivatives
- **1 pts** wrong Q derivative
- **3 pts** wrong approach
- **1 pts** why?
- **1 pts** wrong P derivative

4.2 integration factor **4 / 4**

✓ **- 0 pts Correct**

- **1 pts** sign mistake
- **3 pts** only formula
- **1 pts** a=? b=?
- **4 pts** wrong/no work
- **2 pts** right start

QUESTION 5

SA 6 pts

5.1 dir field **4 / 4**

- **2 pts** No 2. solution
- **2 pts** No 1. solution
- **1 pts** mistake 1. solution
- **1 pts** mistake 2. solution
- **4 pts** doesn't go through the right points
- **2 pts** doesn't go through the right point 1. solution

✓ **+ 4 pts correct**

5.2 Y/N **2 / 2**

- **0.5 pts** 1 incorrect
- **1 pts** 2 incorrect
- **1.5 pts** 3 incorrect
- **2 pts** all incorrect

✓ **+ 2 pts correct**

MIDTERM 1

10/24/2018

Math33B
Nadja Hempel
nadja@math.ucla.edu

Name:

UID:

section:

Problem	Points	Score
1	8	
2	12	
3	7	
4	7	
SA	6	
Total	40	

Exercise 1. (8pt)
Consider the differential equations

$$2y^2 + 4x^2 + 2xy\frac{dy}{dx} = 0$$

(1) Find the integrating factor for the above equations.(4pt)
(Hint: it only depends on x)

$\underbrace{(2y^2+4x^2)}_{P(x,y)}dx + \underbrace{2xy}_{Q(x,y)}dy = 0$

$\frac{\partial P}{\partial y} = 4y, \quad \frac{\partial Q}{\partial x} = 2y,$

Let $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{2xy}(4y - 2y) = \frac{2y}{2xy} = \frac{1}{x}.$

Let the integrating factor $\mu = e^{\int h(x)dx} = e^{\int \frac{dx}{x}} = e^{\ln|x|} = \boxed{|x|.}$

(2) Solve the equation.(4pt)

assuming $x > 0$, $\mu = x$.

$\mu P(x,y)dx + \mu Q(x,y)dy = 0.$

$(2xy^2 + 4x^3)dx + (2x^2y)dy = 0.$

Let $F(x,y) = \int 2x^2y\,dy + \varphi(x) = x^2y^2 + \varphi(x).$

$\frac{\partial F}{\partial x} = 2xy^2 + \varphi'(x) = 2xy^2 + 4x^3 = \mu P(x,y).$

$\varphi'(x) = 4x^3$

$\varphi(x) = x^4.$

$F(x,y) = \boxed{x^2y^2 + x^4 = C}$ implicitly defines a solution to the given ODE.

check: $2xy^2 + 2x^2y\frac{dy}{dx} + 4x^3 = 0$

$2y^2 + 4x^2 + 2xy\frac{dy}{dx} = 0$ ✓

Exercise 2. (12pt) Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2 - y}{x}$$

(1) Find the explicit general solution. (5pt)

Separable: $\frac{dy}{y^2-y} = \frac{dx}{x}$

$\int \frac{dy}{y^2-y} = \int \frac{dx}{x}$

$\int \frac{dy}{y(y-1)} = \ln|x| + C_0$

$(*) \Rightarrow \int \frac{dy}{y-1} - \int \frac{dy}{y} = \ln|x| + C_0$

$\ln\left|\frac{y-1}{y}\right| = \ln|y-1| - \ln|y| = \ln|x| + C_0$

$\left|\frac{y-1}{y}\right| = C_1|x| \qquad C_1 > 0$

$\frac{y-1}{y} = C_2|x| \qquad C_2 \neq 0$

$1 - \frac{1}{y} = C_2|x|$

$1 - C_2|x| = \frac{1}{y}$

$$\boxed{y(x) = \frac{1}{1 - C_2|x|}}$$

$\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y} \quad (*)$

$= \frac{y - (y-1)}{y(y-1)} = \frac{1}{y(y-1)}$.

Check: $\frac{dy}{dx} = \frac{C_2}{(1-C_2x)^2}$

$\frac{y^2-y}{x} = \frac{\frac{1-(1-C_2x)}{(1-C_2x)^2}}{x} = \frac{C_2x}{(1-C_2x)^2} \cdot \frac{1}{x} = \frac{C_2}{(1-C_2x)^2}$

(2) Find the solution to this equation that satisfies the initial condition $y(1) = 2$. (2pt)

$y(1) = \frac{1}{1-C_2} = 2$

$1 = 2(1-C_2)$

$\frac{1}{2} = 1 - C_2$

$C_2 = \frac{1}{2}$.

$$\boxed{y(x) = \frac{1}{1 - \frac{1}{2}|x|}}$$

(3) What is the interval of existence of the solution you found in (b). (3pt)

Finding boundary:

$$1 - \left|\frac{x}{2}\right| = 0$$

$$1 = \left|\frac{x}{2}\right|$$

$$x = \pm 2$$

Interval of existence: $\boxed{(-2, 2)}$

(4) Find the solution to this equation that satisfies the initial condition $y(1) = 0$. (2pt)

We see that the general solution we found in (1) can never equal 0. Thus we go back to orgiral ODE.

$$\frac{dy}{dx} = \frac{y^2 - y}{x} \qquad x = 1,\ y = 0.$$

$$\frac{dy}{dx} = \frac{0}{1} = 0,$$

$$y = \int 0\,dx = C = 0 \Rightarrow C = 0.$$

$$\boxed{y(x) = 0}$$

Exercise 3. (7pt) Suppose there is a tank filled with 100 gallons of water. Pure acid flows into the tank at a rate of 4 gal/min and the well mixed solution leaves the tank at the of 2 gal/min rate. Let $x(t)$ be the volume in gallons of acid in the tank at time t. Find $x(t)$ for any given time t.

Let $v(t)$ represent the total volume of liquid in the tank, at time t.

$$v(t) = 100 + (-4+2)t = 100 - 2t.$$

$$\frac{dx}{dt} = 4 - 2 \cdot \frac{x(t)}{v(t)} = 4 - 2\frac{x}{100-2t}. \qquad x(0) = 0.$$

$$x' + \frac{2}{100-2t}x = 4.$$

$$x' + \frac{1}{50-t}x = 4.$$

$$\mu = e^{\int \frac{dt}{50-t}} = e^{-\ln|50-t|} = \frac{1}{|50-t|}.$$

now, $v(t) > 1 \Rightarrow 100 - 2x > 0$, $t < 50$.
(assuming tank will not be empty)

$$= \frac{1}{50-t}.$$

$$\frac{1}{50-t}x' + \frac{1}{(50-t)^2}x = 4\mu$$

$$(\mu x)' = \frac{4}{50-t}$$

$$\frac{x}{50-t} = \int \frac{4}{50-t}\,dt$$

$$\frac{x}{50-t} = -4\ln\underbrace{|50-t|}_{>0} + C_0$$

$$x = -4(50-t)\ln(50-t) + C_1(50-t)$$

$$x(0) = 0.$$

$$0 = -4 \cdot 50 \ln(50) + C_1 \cdot 50$$

$$C_1 = 4\ln(50).$$

$$\boxed{x(t) = -4(50-t)\ln(50-t) + 4\ln(50)(50-t)}$$

Exercise 4. (7pt) Consider

$$4yxdx + 5x^2dy = 0$$

(1) Show that the above equation is not exact. (3pt)

$\frac{\partial}{\partial y} 4yx = 4x.$

$\frac{\partial}{\partial x} 5x^2 = 10x.$ $\boxed{\neq}$ $\Rightarrow$ not exact.

(2) Find a and b such that $x^a y^b$ is an integration factor of the above equation. (4pt)

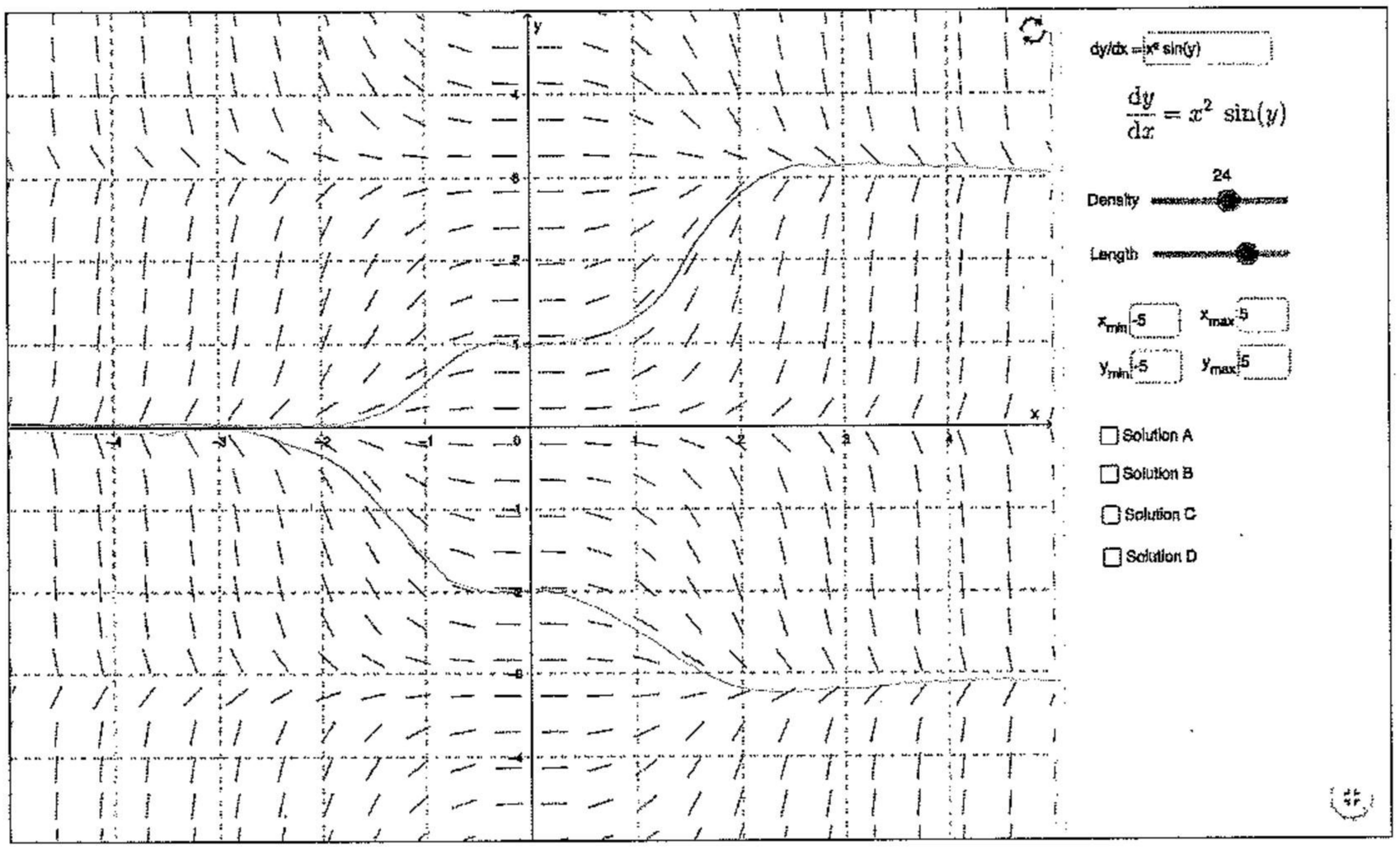
$$\frac{\partial}{\partial y} \underbrace{x^a y^b (4yx)}_{4x^{a+1}y^{b+1}} = \frac{\partial}{\partial x} \underbrace{x^a y^b (5x^2)}_{5x^{a+2}y^b}$$

$$4(b+1)x^{a+1}y^b = 5(a+2)x^{a+1}y^b.$$

$$\boxed{4(b+1) = 5(a+2)}.$$

One such $\boxed{(a,b) = (2,4)}$

Field M1 F18.png



1. Short answer problems

(no explanation needed)

(1) (4pt) Consider the above direction field and draw the solution through (0,1) and the solution through (0,-2).

(2) (2pt) Which of the following are homogeneous differential equations?

(Y) / N $\sin(\frac{x}{y})dy + 2dx = 0$

Y / (N) $(xy + x^2)dy + (y^2x - x^2y)dx$

Y / (N) $\sin(xy)dy - \cos(xy)dx$

(Y) / N $\sqrt{x^2y^2 - 4xy^3}dy + x^2dx$