33B midterm 1

TOTAL POINTS

34.5 / 40

QUESTION 1

integration factor 8 pts

1.1 integration factor 4 / 4

✓ **- 0 pts** **Correct**

- 1 pts minor mistake

- 4 pts no work

- 3 pts subtle work, try to find h(x) but equation incorrect

- 2 pts get h(x),but not u(x)

- 2 pts get u(x) but without details; know how to get u(x) but calculate incorrectly

1.2 solve 4 / 4

✓ **- 0 pts** **Correct**

- 1 pts solution should be in form of F(x,y) = c

- 4 pts no work

- 3 pts know need to do partial integration, but incorrect.

- 2 pts correct form F = \phi + xxxx, but \phi incorrect ; or the other way around.

- 1 pts minor mistake

QUESTION 2

separable eon 12 pts

2.1 explicit solution 5 / 5

✓ **+ 1 pts** **Separating the Equation**

✓ **+ 1 pts** **Partial Fractions**

✓ **+ 1 pts** **Computing Integral**

✓ **+ 1 pts** **Log Rule Application**

✓ **+ 1 pts** **Computing Solution**

+ 2 pts Bernoulli Transformation

+ 1 pts Integrating Factor

+ 2 pts Rest of Bernoulli Solution

+ 0 pts No points

Technically correct, but the arbitrariness of C means you can drop the absolute value.

2.2 y(1) = 2 2 / 2

✓ **+ 2 pts** **Correct Answer**

+ 1.5 pts Correct Answer, Wrong Solution

+ 1 pts Knowing the Process

+ 0 pts No points

2.3 interval of existence 1 / 3

+ 1 pts Knowing 0 is not included

+ 1 pts Correct for their function

+ 1 pts Correct

✓ **+ 1 pts** **Knowing 2 is not included.**

+ 0 pts No points

2.4 y(1) = 0 1 / 2

+ 2 pts Correct Answer

✓ **+ 1 pts** **Correct Answer, but on accident**

+ 0 pts No points

QUESTION 3

3 mixing problem 6 / 7

- 1 pts Identifying x'= rate in- rate out, rate in = 4

- 2 pts Identify rate out = x/(50+t)

- 1 pts Find an integrating factor or homogeneous solution

- 2 pts Find the general solution

- 1 pts Incorporate the initial condition.

- 0 pts Correct

- 1 pts Accidentally made equation Homogeneous/ too simple.

✓ **- 1 pts** **Forgot a factor of 2 in rate out.**

QUESTION 4

exact eqn 7 pts

4.1 not exact 3 / 3

✓ **- 0 pts** **Correct**

- 3 pts No answer

- **2 pts** wrong derivatives
- **1 pts** wrong Q derivative
- **3 pts** wrong approach
- **1 pts** why?
- **1 pts** wrong P derivative

4.2 integration factor **4 / 4**

✓ **- 0 pts Correct**
- **1 pts** sign mistake
- **3 pts** only formula
- **1 pts** a=? b=?
- **4 pts** wrong/no work
- **2 pts** Click here to replace this description.

QUESTION 5

SA 6 pts

5.1 dir field **3 / 4**

- **2 pts** No 2. solution
- **2 pts** No 1. solution
- **1 pts** mistake 1. solution

✓ **- 1 pts mistake 2. solution**
- **4 pts** doesn't go through the right points
- **2 pts** doesn't go through the right point 1. solution

\+ **4 pts** correct

5.2 Y/N **1.5 / 2**

✓ **- 0.5 pts 1 incorrect**
- **1 pts** 2 incorrect
- **1.5 pts** 3 incorrect
- **2 pts** all incorrect

\+ **2 pts** correct

Midterm 1

10/24/2018

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Name:

UID:

section:

Problem	Points	Score
1	8	
2	12	
3	7	
4	7	
SA	6	
Total	40	

Exercise 1. (8pt)
Consider the differential equations

$$2y^2 + 4x^2 + 2xy\frac{dy}{dx} = 0$$

(1) Find the integrating factor for the above equations.(4pt)
(Hint: it only depends on x)

$$(\underbrace{2y^2+4x^2}_{P})\,dx + (\underbrace{2xy}_{Q})\,dy = 0$$

we want u s/t $\frac{\partial}{\partial y}(uP) = \frac{\partial}{\partial x}(uQ)$

$$\Rightarrow \frac{\partial u}{\partial y} P + \frac{\partial P}{\partial y} u = \frac{\partial u}{\partial x} Q + \frac{\partial Q}{\partial x} u$$

Since u depends only on x, $\frac{\partial u}{\partial y} = 0$

$$\frac{\partial P}{\partial y} u = \frac{\partial u}{\partial x} Q + \frac{\partial Q}{\partial x} u$$

$$\frac{\partial u}{\partial x} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)u = h(x)\,u$$

$$h(x) = \frac{1}{2xy}(4y - 2y)$$

$$= \frac{2y}{2xy} = \frac{1}{x}$$

$$\int \frac{\partial u}{u} = \int \frac{1}{x}\,dx$$

$$\ln|u| = \ln|x|$$

$$\boxed{|u| = |x|}$$

$$u = x$$

(2) Solve the equation.(4pt)

$$(\underbrace{2y^2|x| + 4|x|^3}_{P_1})\,dx + \underbrace{2xy|x|}_{Q_1}\,dy = 0$$

$$F(x,y) = \int P_1\,dx + \phi(y)$$

$$= \int 2y^2|x| + 4|x|^3\,dx + \phi(y)$$

$$= y^2x^2 + x^4 + \phi(y)$$

$$Q_1 = \frac{\partial F}{\partial y} = 2yx^2 + \phi'(y) = 2xy|x|$$

$$\phi'(y) = 0. \quad \phi(y) = c$$

$$\boxed{F(x,y) = y^2x^2 + x^4 = c}$$ Implicitly defines the solutions.

Exercise 2. (12pt) Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2 - y}{x}$$

(1) Find the explicit general solution. (5pt)

$$\frac{dy}{y^2 - y} = \frac{dx}{x}$$

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

$$1 = A(y-1) + B(y)$$
$$= Ay - A + By$$
$$= (A+B)y - A$$
$$A + B = 0, \quad -A = 1$$
$$A = -1, \quad B = 1$$

$$\frac{1}{y(y-1)} = \frac{-1}{y} + \frac{1}{y-1}$$

$$\frac{1}{y(y-1)}\,dy = \frac{1}{x}\,dx$$

$$\int\left(\frac{-1}{y} + \frac{1}{y-1}\right)dy = \int \frac{1}{x}\,dx$$

$$-\ln|y| + \ln|y-1| = \ln|x| + C_0$$
$$\ln\left|\frac{y-1}{y}\right| = \ln|x| + C_0$$

Exponentiating: $\frac{y-1}{y} = e^{\ln|x|}e^{C_0} = C_1|x|$

$$-\frac{1}{y} = C_1|x| - 1$$
$$\frac{1}{y} = 1 - C_1|x|$$

$$\boxed{y(x) = \frac{1}{1 - C_1|x|}}$$

(2) Find the solution to this equation that satisfies the initial condition $y(1) = 2$. (2pt)

$$y(1) = \frac{1}{1 - C_1(1)} = 2; \quad C_1 = 1/2.$$

$$y(x) = \frac{1}{1 - \frac{1}{2}|x|} = \frac{2}{2 - |x|}$$

$$\boxed{y(x) = \frac{2}{2 - |x|}}$$

(3) What is the interval of existence of the solution you found in (b). (3pt)

The solution exists for $|x| < 1$,

So since it exists only where $2 - |x| > 0$,

The interval of existence is

$(-2, 2)$

(4) Find the solution to this equation that satisfies the initial condition $y(1) = 0$. (2pt)

$y(x) = \frac{2}{2 - c|x|}$ $y(0) = \frac{2}{2 - c|1|} = \frac{2}{2 - c}$

This solution only occurs when c approaches ∞.

$y(0) = 0$ when $c \to \infty$, so $y(x) = \lim_{c \to \infty} \frac{2}{2 - c(x)} = \underline{0}$

$\boxed{y(x) = 0}$

Exercise 3. (7pt) Suppose there is a tank filled with 100 gallons of water. Pure acid flows into the tank at a rate of 4 gal/min and the well mixed solution leaves the tank at the of 2 gal/min rate. Let $x(t)$ be the volume in gallons of acid in the tank at time t. Find $x(t)$ for any given time t.

4 gal/min Acid

$\boxed{\begin{matrix}100\text{ gal}\\ H_2O\end{matrix}}$

2 gal/min
Solution

$$x'(t) = \text{rate in} - \text{rate out}$$
$$= \frac{4\text{ gal}}{\text{min}} - \frac{x(t)}{100+2t}$$

$$x' = \frac{-1}{100+2t}x + 4$$

using the homogenous solution,

$$x_h' = \frac{-1}{100+2t}x \qquad x_h = e^{-\frac{1}{2}\int\frac{2}{100+2t}dt}$$
$$= e^{-\frac{1}{2}\ln|100+2t|}$$
$$= e^{\ln|(100+2t)^{-1/2}|}$$
$$= |(100+2t)^{-1/2}|$$

since $t \geq 0$, $|(100+2t)^{-1/2}| = (100+2t)^{1/2}$

$$x = v x_h$$

$$v' = \frac{f}{x_h} = \frac{4}{|(100+2t)^{-1/2}|}$$

$$\int v' = \int 4(100+2t)^{1/2}$$

$$v = 2\int 2(100+2t)^{1/2}dt$$
$$= \frac{4(100+2t)^{3/2}}{3} + C$$

$$\boxed{x(t) = \frac{4}{3}(100+2t) - \frac{4000}{3(100+2t)^{1/2}}}$$

$$x = x_h v = \frac{1}{(100+2t)^{1/2}}\cdot\left(\frac{4(100+2t)^{3/2}}{3} + C\right)$$

$$x(t) = \frac{4}{3}(100+2t) + \frac{C}{(100+2t)^{1/2}}$$

$$\frac{C}{\cancel{10}} = \frac{-400}{3}$$

$$x(0) = 0 \Rightarrow \frac{4}{3}(100) + \frac{C}{\sqrt{100}} = \frac{400}{3} + \frac{C}{10} = 0 \qquad C = -\frac{4000}{3}$$

Exercise 4. (7pt) Consider

$$\overbrace{4yx\,dx}^{P} + \overbrace{5x^2\,dy}^{Q}$$

(1) Show that the above equation is not exact. (3pt)

$$\frac{\partial P}{\partial y} = 4x \qquad \frac{\partial Q}{\partial x} = 10x$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the equation is not exact.

(2) Find a and b such that $x^a y^b$ is an integration factor of the above equation. (4pt)

$$P_2 = 4y^{b+1}x^{a+1} \qquad Q_2 = 5x^{a+2}y^{b}$$

For the equation $P_2\,dx + Q_2\,dy = 0$ to be exact, $\frac{\partial P_2}{\partial y}$ must equal $\frac{\partial Q_2}{\partial x}$

$$\frac{\partial P_2}{\partial y} = (b+1)4y^b x^{a+1} = \cancel{4y^b x^{a+1} b + 4y^b x^{a+1}}$$

$$\frac{\partial Q_2}{\partial x} = (a+2)5x^{a+1}y^b = \cancel{5x^{a+1}y^b a + 10x^{a+1}y^b}$$

$$4(b+1)\,y^b x^{a+1} = 5(a+2)\,y^b a^{y+1}$$

$$4(b+1) = 5(a+2)$$

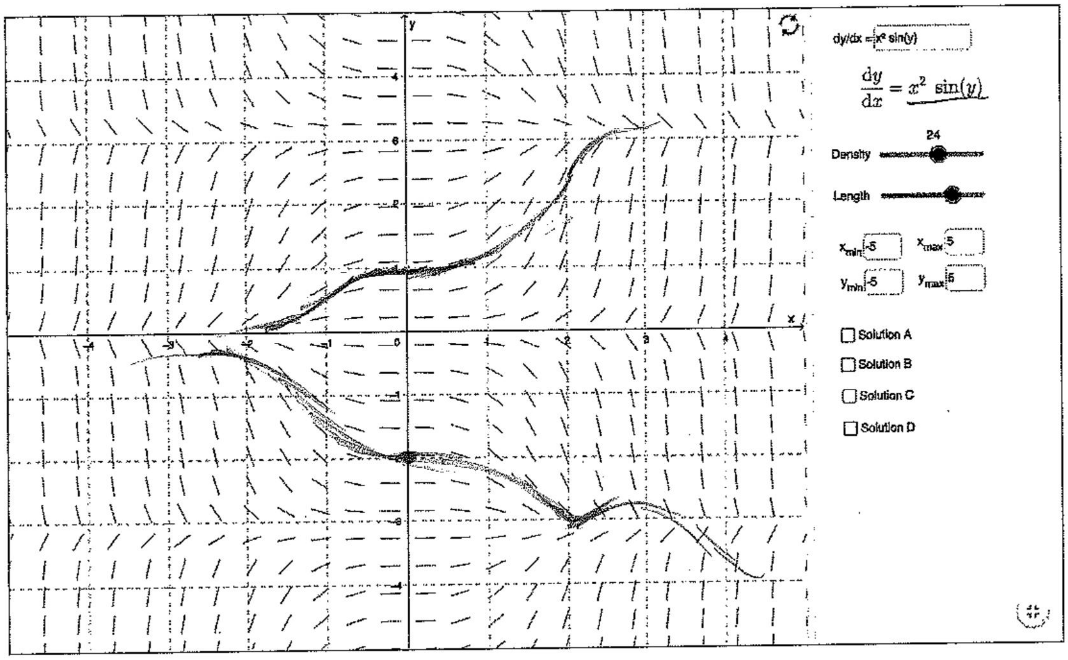
$$4b+4 = 5a+10$$

$$4b-6 = 5a$$

$$a = \frac{4b-6}{5}.$$

For $x^a y^b$ to be an integration factor, a and b must obey the relationship $a = \frac{4b-6}{5}$.

Field M1 F18.png



1. Short answer problems

(no explanation needed)

(1) (4pt) Consider the above direction field and draw the solution through (0,1) and the solution through (0,-2).

(2) (2pt) Which of the following are homogeneous differential equations?

Y / N $\sin(\frac{x}{y})dy + 2dx = 0$ homog, deg 0

Y / N $(xy^{2} + x^2)dy + (y^2x^{3} - x^2y)dx$

Y / N $\sin(xy)dy - \cos(xy)dx$

Y / N $\sqrt{x^2y^2 - 4xy^3}dy + x^2dx$

$t^4 \quad - t^4$

$\sqrt{t^4(x^2y^2) - t^4(4xy^3)}$

$= \sqrt{(t^4 - t^4)(x^2y^2)} = 0\sqrt{x^2y^2}$