

33B Final

Andrew Zheng Kan

TOTAL POINTS

92 / 100

QUESTION 1

1 homogeneous of degree 2 7 / 7

- ✓ - 0 pts Correct
 - 2 pts multiple computational mistakes
 - 5 pts correct integrating factor, but...
 - 2 pts didn't give the explicit solution for y
 - 1 pts computation error
 - 7 pts wrong answer

QUESTION 2

second order 13 pts

2.1 general homogeneous 2 / 2

- ✓ + 1 pts Correct roots of characteristic polynomial
- ✓ + 1 pts Correct general solution given chosen roots
- + 0 pts Incorrect / minimal progress

2.2 particular 2 / 2

- ✓ - 0 pts Correct (given answer to part 1)
 - 1 pts Minor mistake in setup
 - 2 pts Incorrect / minimal progress

2.3 ex/uniqueness thé 1 / 2

- 0 pts Correct
- 1 pts Did not get multiple solutions in previous part, but made no mention that the initial data was given at different points
- ✓ - 1 pts Stating that uniqueness doesn't apply, but does not give correct reason why
 - 2 pts Incorrect / minimal progress

2.4 particular (poly) 2 / 2

- ✓ + 2 pts Correct
 - + 1 pts Attempting method of undetermined coefficients
 - + 0 pts Incorrect / minimal progress

2.5 particular (cos) 4 / 4

- ✓ + 4 pts Correct
 - + 2 pts Identifying by some means that we're in the "exceptional case"
 - + 1 pts Attempting method of undetermined coefficients
 - + 0 pts Incorrect / minimal progress

2.6 general inhomogeneous 1 / 1

- ✓ - 0 pts Correct (given previous work)
- 1 pts Incorrect / minimal progress

QUESTION 3

3 linear system 10 / 10

- ✓ + 1 pts eigenvalues
- ✓ + 1 pts eigenvector -1
- ✓ + 1 pts eigenvector 2
- ✓ + 1 pts squaring matrix (A-2I)
- ✓ + 1 pts generalized eigenvector 2
- ✓ + 2 pts solutions to 2
- ✓ + 1 pts general solution
 - + 1 pts error in initial value
 - + 0 pts incorrect/no solution

QUESTION 4

second order 3 pts

4.1 solutions 2 / 2

- ✓ - 0 pts Correct
- 2 pts Minimal progress

4.2 solution ass. homogeneous 0 / 1

- 0 pts Correct
- ✓ - 1 pts Incorrect / minimal progress

QUESTION 5

linear system 9 pts

5.1 general solution 6 / 6

- ✓ + 6 pts Correct
 - + 1 pts Correct characteristic polynomial
 - + 1 pts Correct eigenvalues given characteristic polynomial
 - + 1 pts Valid setup to find eigenvector
 - + 1 pts Correct eigenvector
 - + 1 pts Writing out the complex-valued solution $e^{(\lambda t)v}$, or using a formula to skip this step
 - + 1 pts Final real-valued solution given previous work
- + 0 pts Incorrect / minimal progress

5.2 spiral 1 / 1

- ✓ - 0 pts Correct
 - 1 pts Did not specify that $\text{Re}(\lambda)$ not equal 0 or check $T^2 - 4D < 0$
- 1 pts Incorrect / skipped

5.3 sink/source 1 / 1

- ✓ - 0 pts Correct
 - 1 pts Incorrect / skipped

5.4 direction 1 / 1

- ✓ - 0 pts Correct
 - 1 pts Incorrect / skipped

QUESTION 6

autonomous 8 pts

6.1 phase line 3 / 3

- ✓ + 3 pts Correct
 - + 1 pts Correct equilibrium points
 - + 1 pts Correct graph or derivative test
 - + 1 pts Correct phase line arrows given graph
- + 0 pts Incorrect / minimal progress

6.2 sketch 3 / 3

- ✓ + 3 pts Correct (given previous work)

- + 1 pts Equilibrium lines
- + 1 pts Identifying stability of equilibrium lines
- + 1 pts Sketch of solutions in each region given previous work
- + 0 pts Incorrect / minimal progress

6.3 exists?? 2 / 2

- ✓ + 1 pts Correct conclusion (no)
- ✓ + 1 pts Correct reasoning ($x(0)=1$ corresponds to equilibrium solution $x(t)=1$; apply uniqueness)
- + 0 pts Incorrect / minimal progress

QUESTION 7

Existence uniqueness 5 pts

7.1 $x_0(0)=2$ 1 / 3

- 0 pts Correct
- ✓ - 1 pts no/wrong partial derivative, i.e., no uniqueness theorem
- ✓ - 1 pts no correct interval for x
 - 1 pts no correct interval for t
 - 1 pts not for that specific point
- 3 pts wrong answer

7.2 $x_0(1)=1$? 2 / 2

- ✓ - 0 pts Correct
 - 1 pts wrong reason
- 2 pts wrong answer

QUESTION 8

8 Matrices 8 / 9

- ✓ + 1 pts A Nodal
- ✓ + 1 pts A source
- ✓ + 1 pts A picture
 - + 1 pts B star
- ✓ + 1 pts B sink
- ✓ + 1 pts B picture
- ✓ + 1 pts C degenerate node
- ✓ + 1 pts C source
- ✓ + 1 pts D saddle
 - + 0 pts everything wrong/no solution

QUESTION 9

9.3. order -> linear system 3 / 3

- ✓ + 1 pts substitution $v_1=y'$
- ✓ + 1 pts substitution $v_2=v_1'=y''$
- ✓ + 1 pts right form
- + 0 pts wrong/no solution

QUESTION 10

10 roast in oven 8 / 8

- ✓ - 0 pts Correct
- 4 pts correct formula, but wrong after.
- 6 pts correct setting, but wrong integration
- 1 pts computational miss
- 2 pts multiple computational mistakes

QUESTION 11

11 mixing problem 9 / 9

- ✓ - 0 pts Correct
- 1 pts 1 rate in/out incorrect
- 2 pts 2 rates in/out incorrect
- 2 pts initial value incorrect
- 1 pts wrongly written in matrix
- 7 pts wrong matrix/no matrix
- + 1 pts rate in - rate out

QUESTION 12

12 integrating factor 4 / 4

- ✓ + 4 pts Correct
- + 1 pts Writing exactness condition
- + 1 pts Computing correct derivatives
- + 1 pts $a=3$
- + 1 pts $b=2$
- + 0 pts Skipped / minimal progress

QUESTION 13

2. order 12 pts

13.1 fundamental set of solutions 4 / 6

- 0 pts Correct
- ✓ - 2 pts Do not check linearly independency
- 1 pts one computation error
- 2 pts -2 two computation errors

- 3 pts Three computation errors

- 6 pts wrong answer

13.2 variation of parameters 5 / 6

- 0 pts Correct
- ✓ - 1 pts a computational error
- 2 pts multiple computational errors
- 1 pts wrong general solution
- 4 pts do not remember formulas

FINAL

3/22/2019

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section: 210

Math33B

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Problem	Points	Score
1	7	
2	13	
3	10	
4	3	
5	9	
6	8	
7	5	
8	9	
9	3	
10	8	
11	9	
12	4	
13	12	
Total	100	

Instructions

- (1) Enter your name, SID number, and discussion section on the top of this page.
- (2) If you need **more space**, use the extra page at the end of the exam.
- (3) NO Calculators, computers, books or notes of any kind are allowed.
- (4) Show your work. Unsupported answers will receive few or no credit.
- (5) Good Luck!

Exercise 1. (7pt) Consider the differential equation:

$$(x^2 + y^2) dx - 2xy dy = 0.$$

(1) Give an implicit solution (i.e. of the form $F(x, y) = C$). (5pt)

(2) Give the explicit solution for $y(x)$. (2pt)

(Hint: Find an integrating factor)

(1)

$$P dx + Q dy = 0$$

$$\frac{dP}{dy} = 2y, \quad \frac{dQ}{dx} = -2y$$

$$h(x) = \frac{1}{Q} \left(\frac{dP}{dy} - \frac{dQ}{dx} \right)$$

$$= \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\mu = e^{\int h(x) dx} = e^{-2 \ln|x|} = \frac{1}{x^2}$$

$$\mu (P dx + Q dy)$$

$$= \left(1 + \frac{y^2}{x^2} \right) dx - 2 \frac{y}{x} dy = 0$$

$$F(x, y) = \int \left(1 + \frac{y^2}{x^2} \right) dx + \phi(y)$$

$$-\frac{2y}{x} = \frac{d}{dy} \int \left(1 + \frac{y^2}{x^2} \right) dx + \phi'(y)$$

$$= \frac{d}{dy} \left(x - \frac{y^2}{x} \right) + \phi'(y)$$

$$= -\frac{2y}{x} + \phi'(y)$$

$$\phi'(y) = 0 \Rightarrow \phi(y) = 0$$

$$\boxed{x - \frac{y^2}{x} = C}$$

(2)

$$y^2 = x^2 - xC$$

$$C_1 = -C$$

$$\boxed{y(x) = \sqrt{x^2 + xC_1}}$$

Exercise 2. (13pt)

(1) Find the real valued general solution of $y'' + 9y = 0$. (2pt)

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm 3i$$

$$y(t) = C_1 e^{0t} \cos 3t + C_2 e^{0 \cdot t} \sin 3t$$

$$y(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

(2) Find all the solutions to the initial value problem $y(\pi/6) = 0$, $y'(0) = 0$. (2pt)

$$y(\pi/6) = 0$$

$$C_1 \cos(\pi/2) + C_2 \sin(\pi/2) = 0$$

$$C_2 = 0$$

$$y'(0) = 0$$

$$-3C_1 \sin(3t) + 3C_2 \cos(3t) = 0$$

$$-3C_1 = 0$$

$$C_1 = 0$$

$$y(t) = C_1 \cos(3t), C_1 \in \mathbb{R}$$

- (3) Why does the previous part not violate the 2. order existence and uniqueness theorem? (2pt)

Since y' is not continuous and defined from $t=0$ to $t=\pi/6$, the uniqueness theorem doesn't hold

- (4) Find a particular solution of

$$y'' + 9y = -9t^2 + 16 \quad (2pt)$$

$$y_p = At^2 + Bt + C$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

$$2A + 9(At^2 + Bt + C) = -9t^2 + 16$$

$$9A = -9$$

$$A = -1$$

$$9B = 0$$

$$B = 0$$

$$2A + 9C = 16$$

$$C = 2$$

$$y_p(t) = -t^2 + 2$$

(5) Find a particular solution of

$$y'' + 9y = 12 \cos(-3t) \quad (4pt)$$

$$y_p = A \cos(-3t) + B \sin(-3t)$$

$$y_p' = +3A \sin(-3t) - 3B \cos(-3t)$$

$$y_p'' = -9A \cos(-3t) - 9B \sin(-3t)$$

$$-9A \cos(-3t) - 9B \sin(-3t) + 9(A \cos(-3t) + B \sin(-3t)) = 0$$

$$y_p = At \cos(-3t) + Bt \sin(-3t)$$

$$y_p' = +3At \sin(-3t) + A \cos(-3t) - 3Bt \cos(-3t) + B \sin(-3t)$$

$$y_p'' = -9At \cos(-3t) + 3A \sin(-3t) + 3A \sin(-3t) - 9Bt \sin(-3t) - 3B \cos(-3t) - 3B \cos(-3t) - 3B \cos(-3t) = -9t(B \sin(-3t) + A \cos(-3t)) + 6(A \sin(-3t) - B \cos(-3t))$$

$$y'' + 9y = 6A \sin(-3t) - 6B \cos(-3t) = 12 \cos(-3t)$$

$$A = 0, B = -2$$

$$y_p = -2t \sin(-3t)$$

(6) Give the general solution to the following differential equation (Hint: Use Part (1),(4),(5) of this exercise)

$$y'' + 9y = 18t^2 - 32 - 24 \cos(-3t) \quad (1pt)$$

$$y(t) = (C_1 \cos(3t) - 2(-t^2 + 2) - 2(-2t \sin(-3t)))$$

$$= C_1 \cos(3t) + 2t^2 - 4 + 4t \sin(-3t)$$

Exercise 3. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} 2 & -3 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = -1, 2, 2, 2$$

$$\lambda = -1 \quad \begin{pmatrix} 3 & -3 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 0 & -3 & 1 & 4 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -3 & 1 & 4 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -3 & 1 & 4 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_4 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$y(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_4 e^{2t} \left[\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

$$t=0, \quad \begin{aligned} c_1 + c_2 - 3c_3 &= 4 \\ c_1 + c_3 + c_4 &= 2 \\ 3c_3 + 4c_4 &= 0 \\ -1 \cdot c_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_4 &= 0 \\ c_3 &= 0 \\ c_1 &= 2 \\ c_2 &= 2 \end{aligned}$$

(you may use the next page for additional computations)

$$y_p(t) = 2e^{-t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise 4. (3pt) Consider the differential equation

$$\frac{y''}{e^x} - y' + y = 1.$$

(1) Show that $y_1(x) = e^x$ and $y_2(x) = 1$ are two solutions. (2pt)

$$y_1' = e^x \quad y_1'' = e^x$$

$$\frac{e^x}{e^x} - e^x + e^x = 1$$

$$1 = 1 \quad \checkmark$$

$$y_2' = 0, \quad y_2'' = 0$$

$$\frac{0}{e^x} - 0 + 1 = 1 \quad \checkmark$$

(2) Using (1), find a solution to the associated homogenous equation. (1pt)

$$W_{y_1, y_2} = e^x \cdot 1 - 0 \cdot e^x \neq 0$$

$$y(x) = (c_1 e^x + c_2 \cdot 1)$$

Exercise 5. (9pt) Let

$$A = \begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (6pt)

$$\begin{pmatrix} 4-\lambda & 5 \\ -1 & 2-\lambda \end{pmatrix}$$

$$(4-\lambda)(2-\lambda) + 5 = 0$$

$$\lambda^2 - 6\lambda + 13 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-52}}{2}$$

$$= 3 \pm 2i$$

$$\lambda = 3+2i: \begin{pmatrix} 4-(3+2i) & 5 \\ -1 & 2-(3+2i) \end{pmatrix} = \begin{pmatrix} 1-2i & 5 \\ -1 & -1-2i \end{pmatrix}$$

$$\begin{pmatrix} 1-2i & 5 \\ -1 & -1-2i \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$$

$$e^{(3+2i)t} \begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$$

$$= e^{3t} (\cos(2t) + i\sin(2t)) \begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$$

$$= e^{3t} \begin{pmatrix} \cos(2t) + 2i\cos(2t) + i\sin(2t) - 2\sin(2t) \\ -\cos(2t) - i\sin(2t) \end{pmatrix}$$

$$y(t) = c_1 e^{3t} \begin{pmatrix} \cos(2t) + 2\sin(2t) \\ -\cos(2t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2\cos(2t) + \sin(2t) \\ -\sin(2t) \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix}$$

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(2) Conclude that the equilibrium point is a spiral. (1pt)

$$T = 6, \quad D = 8 + 5 = 13$$

$$T^2 - 4D < 0$$

$$T \neq 0$$

→ spiral

(3) Is it a sink or a source? (1pt)

$$T > 0$$

→ source

(4) Does the spiral rotate clockwise or counterclockwise? (1pt)

$$a_{21} = -1 < 0$$

→ clockwise

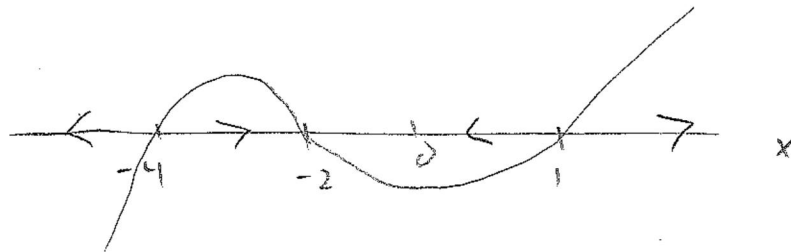
Exercise 6. (8pt)

Consider the differential equation

$$\frac{dx}{dt} = e^{-x}(x^3 + 5x^2 + 2x - 8)$$

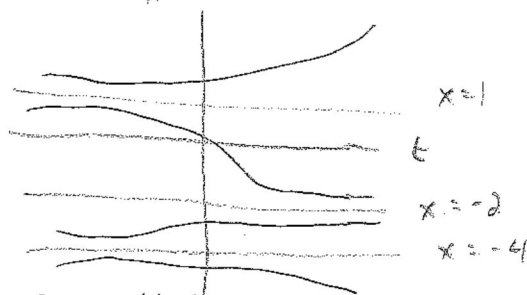
- (1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

$$\begin{aligned} x=1, & \quad 1^3 + 5 + 2 - 8 = 0 \quad \checkmark \\ x=-4, & \quad -64 + 80 - 8 - 8 = 0 \quad \checkmark \\ x=-2, & \quad -8 + 20 - 4 - 8 = 0 \quad \checkmark \end{aligned}$$



- (2) Sketch the equilibrium points on the tx -plane and identify the stable and unstable points. The equilibrium solutions divide the tx -plane into regions. Sketch at least one solution curve in each of these regions. (3pt)

$x = 1, -4$ are unstable, $x = -2$ is stable



- (3) Does there exist a solution $x(t)$ of the equation satisfying $x(0) = 1$ and $x(1) = 0$? (1pt) Justify your answer. (1pt)

No, $x=1$ is an equilibrium solution and no line from $x(1)=0$ can touch it

Exercise 7. (5pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x-2}{(t^2-4)(-x^2+3x)}$$

- (1) Consider the initial value problem $x_0(0) = 2$. Find the biggest rectangle around this point for which you can apply the existence and the uniqueness theorem. (3pt)

$$t \neq 2, -2, x \neq 0, x \neq -3$$

$$t=0, x=2$$

$$\boxed{(-2, 2) \times (0, \infty)}$$

- (2) Can $x_0(1) = 1$? (1pt) Justify your answer. (1pt)

$$x=1, t=1 \quad \frac{dx}{dt} = \frac{1-2}{(1-4)(-1+3)} = \frac{-1}{-3 \cdot 2} = \frac{1}{6}$$

$$\text{no, } \frac{dx_0}{dt} > 0 \text{ for } t \in (0, 1), x \in (1, 2)$$

$$\text{and } x_0(1) < x_0(0)$$

Exercise 8. (9pt) Let

$$A = \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

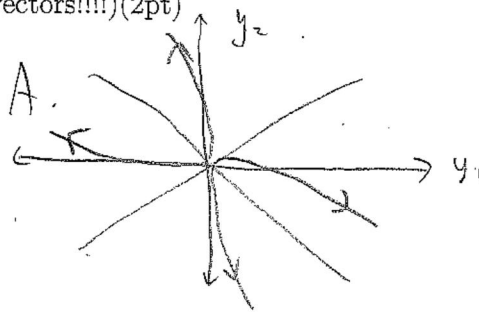
$$C = \begin{pmatrix} 2 & 1 \\ -4 & 6 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

- (1) For each of the matrices above decide whether they are a saddle, nodal sink/source, degenerated nodal sink/source, star (sink/source) (7pt)
 (2) For A and B draw a **rough** sketch of the half line solutions. (You do NOT need to determine the actual eigenvectors!!!!)(2pt)

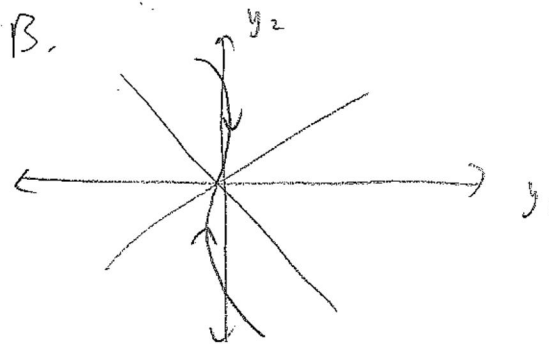
A. $T = 7, D = 10$
 $T^2 - 4D > 0$
 $T > 0, D > 0$

nodal source



B. $T = -2, D = 1$
 $T^2 - 4D = 0$
 $T < 0$

degenerate nodal sink



C. $T = 8, D = 16$
 $T^2 - 4D = 0$

$$T > 0$$

degenerate nodal source

D. $T = 0, D = -4$
 $T^2 - 4D > 0$
 $T = 0$
 $D < 0$

saddle

Exercise 9. (3pt)

Consider the third order equation $y''' + 3 \cos(2t)y'' - 2y' - (t^2 + 1)y = -3e^t$. Write this equations as a planar system of first-order equations.

$$y' = y_1$$

$$y_1' = y_2$$

$$y_2' = -3 \cos(2t)y_2 + 2y_1 + (t^2 + 1)y - 3e^t$$

Exercise 10. (8pt)

A roast is removed from the oven when its temperature is 270, and 10 minutes later its temperature is 170. Assume Newton's law of cooling, and a room temperature is 70. How much (after removing it from the oven) should you allow the roast to cool, if you want to serve it when its temperature is 150? Since calculators are not allowed, leave your answer in logarithms. Remember that Newton's law of cooling says $dT/dt = k(T - A)$, where T is temperature, t is time, A is surrounding temperature and k is a constant.

$$\frac{dT}{dt} = k(T - 70)$$

$$\int \frac{dT}{T-70} = \int k dt$$

$$\ln |T-70| = kt$$

$$T = Ae^{kt} + 70$$

$$T(0) = 270 \rightarrow A = 200$$

$$T(10) = 170,$$

$$170 = 200e^{10k} + 70$$

$$e^{10k} = \frac{1}{2}$$

$$k = \frac{\ln(1/2)}{10} = -\frac{\ln 2}{10}$$

$$T = 200e^{-\frac{\ln 2}{10}t} + 70$$

$$150 = 200e^{-\frac{\ln 2}{10}t} + 70$$

$$e^{-\frac{\ln 2}{10}t} = \frac{2}{5}$$

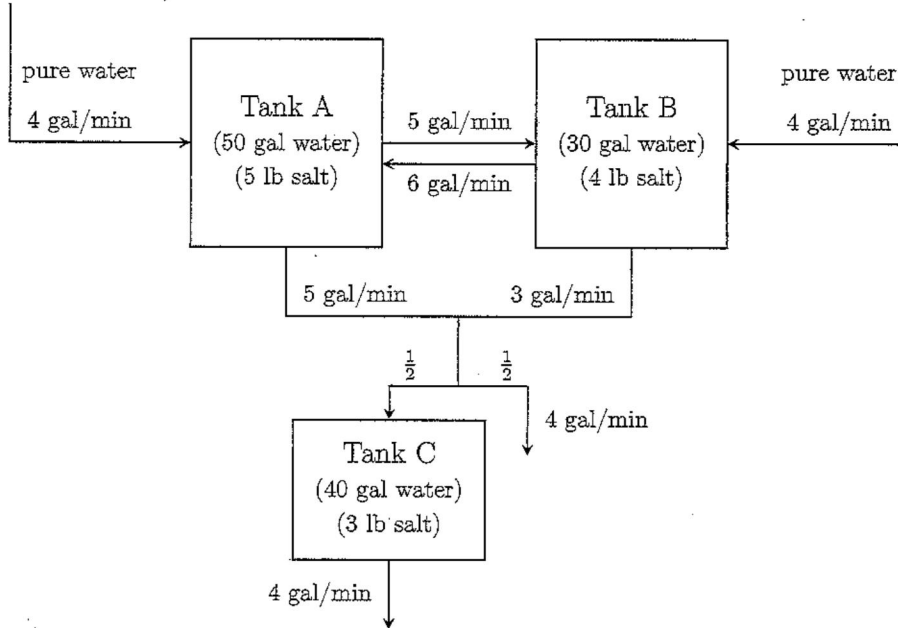
$$-\frac{\ln 2}{10} \cdot t = \ln \frac{2}{5}$$

(you may use the next page for additional computations)

$$t = -\frac{10}{\ln 2} (\ln 2 - \ln 5)$$

$$= \boxed{-10 + 10 \frac{\ln 5}{\ln 2}}$$

Exercise 11. (9pt) Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ (in lb) in tank A, B, and C at time t (you do NOT have to solve it!).



Please write your final solution below (in particular write it as a matrix equation):

$$\begin{pmatrix} x'_A \\ x'_B \\ x'_C \end{pmatrix} = \begin{pmatrix} -1/5 & 1/5 & 0 \\ 1/10 & -3/10 & 0 \\ 1/20 & 1/20 & -1/10 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \quad \begin{pmatrix} x_A(0) \\ x_B(0) \\ x_C(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

(you may use the next page for additional computations)

$$\frac{dx_A}{dt} = -\frac{10x_A}{50} + \frac{6}{30}x_B = -\frac{1}{5}x_A + \frac{1}{5}x_B$$

$$\frac{dx_B}{dt} = -\frac{9x_B}{30} + \frac{5}{50}x_A = -\frac{3}{10}x_B + \frac{1}{10}x_A$$

$$\begin{aligned} \frac{dx_C}{dt} &= -\frac{4}{40}x_C + \frac{1}{2} \left(\frac{5x_A}{50} + \frac{3x_B}{30} \right) \\ &= -\frac{1}{10}x_C + \frac{x_A}{20} + \frac{x_B}{20} \end{aligned}$$

Exercise 12. (4pt) Consider the following differential equation:

$$4y dx + x(1 + 2y^2) dy$$

Find integers a and b such that $x^a e^{y^b}$ is an integration factor of the above equation.

$$\left(\underbrace{4y x^{+a} e^{y^b}}_P dx + \underbrace{e^{-y^b} x^{1+a} (1+2y^2)}_Q dy \right) = 0$$

$$\begin{aligned} \frac{dP}{dy} &= x^{+a} (4e^{y^b} + 4y \cdot b y^{b-1} e^{y^b}) \\ &= x^{+a} (4e^{y^b} + 4b y^b e^{y^b}) \end{aligned}$$

$$\frac{dQ}{dx} = (1+a) x^{1+a-1} (1+2y^2) e^{y^b}$$

$$\frac{dP}{dy} = \frac{dQ}{dx}$$

$$x^a (4e^{y^b} + 4b y^b e^{y^b}) = (1+a) x^a (1+2y^2) e^{y^b}$$

$$4 + 4b y^b = (1+a) (1+2y^2)$$

$$1 = (1+a) + 2y^2 / (1+a)$$

$$\boxed{b=2}$$

$$4 = 1+a$$

$$\boxed{a=3}$$

Exercise 13. (12pt) Consider the following differential equation:

$$y'' - \frac{6}{t}y' + \frac{10}{t^2}y = 12t^4$$

- (1) Decide which of following functions form a fundamental set of solutions to the associated homogeneous equation: (6pt)

$$t^2, t^3, t^4, t^5, t^6$$

$$y = t^2, \quad y' = 2t, \quad y'' = 2$$

$$2 - \frac{6}{t} \cdot 2t + \frac{10}{t^2} \cdot t^2 = 0 \quad \checkmark$$

$$y = t^3, \quad y' = 3t^2, \quad y'' = 6t$$

$$6t - \frac{6}{t} \cdot 3t^2 + \frac{10}{t^2} \cdot t^3 = -2t \neq 0 \quad \times$$

$$y = t^4, \quad y' = 4t^3, \quad y'' = 12t^2$$

$$12t^2 - \frac{6}{t} \cdot 4t^3 + \frac{10}{t^2} \cdot t^4 = -2t^4 \neq 0 \quad \times$$

$$y = t^5, \quad y' = 5t^4, \quad y'' = 20t^3$$

$$20t^3 - \frac{6}{t} \cdot 5t^4 + \frac{10}{t^2} \cdot t^5 = 0 \quad \checkmark$$

$$y = t^6, \quad y' = 6t^5, \quad y'' = 30t^4$$

$$30t^4 - \frac{6}{t} \cdot 6t^5 + \frac{10}{t^2} \cdot t^6 = 4t^4 \neq 0 \quad \times$$

$$\boxed{t^2 \ \& \ t^5}$$

- (2) Using variation of parameter, find the general solution to the inhomogeneous linear 2.order differential equation. (6pt)

$$y_1 = t^2, \quad y_2 = t^5$$

$$W_{y_1, y_2} = t^2 \cdot 5t^4 - t^5 \cdot 2t = 3t^6 \neq 0 \quad \checkmark$$

$$V_1 = \int \frac{t^2 \cdot 12t^4}{3t^6} dt$$

$$= \int 4 dt$$

$$= 4t$$

$$V_2 = \int \frac{-t^5 \cdot 12t^4}{3t^6} dt$$

$$= -4 \int t^3 dt$$

$$= -t^4$$

$$y(t) = V_1 y_1 + V_2 y_2 + C_1 y_1 + C_2 y_2$$

$$= 4t \cdot t^2 + t^5 \cdot (-t^4) + C_1 t^2 + C_2 t^5$$

$$= \boxed{4t^3 - t^9 + C_1 t^2 + C_2 t^5}$$

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