33B Final

Max C Wu

TOTAL POINTS

97 / 100

QUESTION 1

1 homogeneous of degree 2 **7 / 7**

✓ **- 0 pts** Correct

- 2 pts multiple computational mistakes

- 5 pts correct integrating factor, but...

- 2 pts didn't give the explicit solution for y

- 1 pts computation error

- 7 pts wrong answer

QUESTION 2

second order 13 pts

2.1 general homogeneous **2 / 2**

✓ **+ 1 pts** **Correct roots of characteristic polynomial**

✓ **+ 1 pts** **Correct general solution given chosen roots**

+ 0 pts Incorrect / minimal progress

2.2 particular **2 / 2**

✓ **- 0 pts** **Correct (given answer to part 1)**

- 1 pts Minor mistake in setup

- 2 pts Incorrect / minimal progress

2.3 ex/uniqueness thé **2 / 2**

✓ **- 0 pts** Correct

- 1 pts Did not get multiple solutions in previous part, but made no mention that the initial data was given at different points

- 1 pts Stating that uniqueness doesn't apply, but does not give correct reason why

- 2 pts Incorrect / minimal progress

2.4 particular (poly) **2 / 2**

✓ **+ 2 pts** Correct

+ 1 pts Attempting method of undetermined coefficients

+ 0 pts Incorrect / minimal progress

2.5 particular (cos) **4 / 4**

✓ **+ 4 pts** Correct

+ 2 pts Identifying by some means that we're in the "exceptional case"

+ 1 pts Attempting method of undetermined coefficients

+ 0 pts Incorrect / minimal progress

2.6 general inhomogeneous **1 / 1**

✓ **- 0 pts** **Correct (given previous work)**

- 1 pts Incorrect / minimal progress

QUESTION 3

3 linear system **10 / 10**

✓ **+ 1 pts** **eigenvalues**

✓ **+ 1 pts** **eigenvector -1**

✓ **+ 1 pts** **eigenvector 2**

✓ **+ 1 pts** **squaring matrix (A-2I)**

✓ **+ 1 pts** **generalized eigenvector 2**

✓ **+ 2 pts** **solutions to 2**

✓ **+ 1 pts** **general solution**

✓ **+ 2 pts** **initial value correct**

+ 1 pts error in initial value

+ 0 pts incorrect/no solution

QUESTION 4

second order 3 pts

4.1 solutions **2 / 2**

✓ **- 0 pts** Correct

- 2 pts Minimal progress

4.2 solution ass. homogeneous **1 / 1**

✓ **- 0 pts** Correct

- 1 pts Incorrect / minimal progress

QUESTION 5

linear system 9 pts

5.1 general solution 6 / 6

✓ **+ 6 pts** **Correct**

+ 1 pts Correct characteristic polynomial

+ 1 pts Correct eigenvalues given characteristic polynomial

+ 1 pts Valid setup to find eigenvector

+ 1 pts Correct eigenvector

+ 1 pts Writing out the complex-valued solution e^(λt)v, or using a formula to skip this step

+ 1 pts Final real-valued solution given previous work

+ 0 pts Incorrect / minimal progress

5.2 spiral 1 / 1

✓ **- 0 pts** **Correct**

- 1 pts Did not specify that Re(λ) not equal 0 or check T^2-4D<0

- 1 pts Incorrect / skipped

5.3 sink/source 1 / 1

✓ **- 0 pts** **Correct**

- 1 pts Incorrect / skipped

5.4 direction 1 / 1

✓ **- 0 pts** **Correct**

- 1 pts Incorrect / skipped

QUESTION 6

autonomous 8 pts

6.1 phase line 3 / 3

✓ **+ 3 pts** **Correct**

+ 1 pts Correct equilibrium points

+ 1 pts Correct graph or derivative test

+ 1 pts Correct phase line arrows given graph

+ 0 pts Incorrect / minimal progress

6.2 sketch 3 / 3

✓ **+ 3 pts** **Correct (given previous work)**

+ 1 pts Equilibrium lines

+ 1 pts Identifying stability of equilibrium lines

+ 1 pts Sketch of solutions in each region given previous work

+ 0 pts Incorrect / minimal progress

6.3 exists?? 2 / 2

✓ **+ 1 pts** **Correct conclusion (no)**

✓ **+ 1 pts** **Correct reasoning (x(0)=1 corresponds to equilibrium solution x(t)=1; apply uniqueness)**

+ 0 pts Incorrect / minimal progress

QUESTION 7

Existence uniques 5 pts

7.1 x_0(0)=2 3 / 3

✓ **- 0 pts** **Correct**

- 1 pts no/wrong partial derivative, i.e., no uniqueness theorem

- 1 pts no correct interval for x

- 1 pts no correct interval for t

- 1 pts not for that specific point

- 3 pts wrong answer

7.2 x_0(1)=1? 2 / 2

✓ **- 0 pts** **Correct**

- 1 pts wrong reason

- 2 pts wrong answer

QUESTION 8

8 Matrices 8 / 9

✓ **+ 1 pts** **A Nodal**

✓ **+ 1 pts** **A source**

✓ **+ 1 pts** **A picture**

+ 1 pts B star

✓ **+ 1 pts** **B sink**

✓ **+ 1 pts** **B picture**

✓ **+ 1 pts** **C degenerate node**

✓ **+ 1 pts** **C source**

✓ **+ 1 pts** **D sattle**

+ 0 pts everything wrong/no solution

QUESTION 9

9 3. order -> linear system 2 / 3

✓ **+ 1 pts** **substitution v_1=y'**

✓ **+ 1 pts** **substitution v_2=v_1'=y''**

+ 1 pts right form

+ 0 pts wrong/no solution

QUESTION 10

10 roast in oven 8 / 8

✓ **- 0 pts** **Correct**

- 4 pts correct formula, but wrong after.

- 6 pts correct setting, but wrong integration

- 1 pts computational miss

- 2 pts multiple computational mistakes

QUESTION 11

11 mixing problem 9 / 9

✓ **- 0 pts** **Correct**

- 1 pts 1 rate in/out incorrect

- 2 pts 2 rates in/out incorrect

- 2 pts initial value incorrect

- 1 pts wrongly written in matrix

- 7 pts wrong matrix/no matrix

+ 1 pts rate in - rate out

QUESTION 12

12 integrating factor 4 / 4

✓ **+ 4 pts** **Correct**

+ 1 pts Writing exactness condition

+ 1 pts Computing correct derivatives

+ 1 pts a=3

+ 1 pts b=2

+ 0 pts Skipped / minimal progress

QUESTION 13

2. order 12 pts

13.1 fundamental set of solutions 6 / 6

✓ **- 0 pts** **Correct**

- 2 pts Do not check linearly independency

- 1 pts one computation error

- 2 pts -2 two computation errors

- 3 pts Three computation errors

- 6 pts wrong answer

13.2 variation of parameters 5 / 6

- 0 pts Correct

✓ **- 1 pts** **a computational error**

- 2 pts multiple computational errors

- 1 pts wrong general solution

- 4 pts do not remember formulas

70

FINAL

3/22/2019

Name: Max Wu

section: 2B

Math33B
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Problem	Points	Score
1	7	
2	13	
3	10	
4	3	
5	9	
6	8	
7	5	
8	9	
9	3	
10	8	
11	9	
12	4	
13	12	
Total	100	

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Consider the differential equation:

$$(x^2 + y^2)\,dx - 2xy\,dy = 0.$$

(1) Give an implicit solution (i.e. of the form $F(x,y) = C$). (5pt)
(2) Give the **explicit** solution for $y(x)$. (2pt)

(*Hint: Find an integrating factor*)

$P = (x^2+y^2)$

$Q = -2xy$

$h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= \frac{1}{-2xy}(2y+2y) = -\frac{2}{x} \Rightarrow \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$

$\Rightarrow \left(1+\frac{y^2}{x^2}\right)dx - 2\left(\frac{y}{x}\right)dy = 0$

$\frac{\partial P}{\partial y} = \frac{2y}{x^2} = \frac{2y}{x^2} = \frac{\partial Q}{\partial x}$

$P = 1 + \frac{y^2}{x^2}, \quad Q = -\frac{2y}{x}$

$F(x,y) = \int P\,dx + \phi(y) = x - \frac{y^2}{x} + \phi(y)$

$Q = -\frac{2y}{x} = \frac{\partial}{\partial y}\int P\,dx + \phi'(y) = -\frac{2y}{x} + \phi'(y) \Rightarrow \phi'(y) = 0$

$\Rightarrow \phi(y) = c$

$\Rightarrow F(x,y) = \boxed{x - \frac{y^2}{x} = C}$

explicit: $\boxed{y = \pm\sqrt{(x-C)x}}$

Exercise 2. ~~(13pt)~~

(1) Find the **real valued general** solution of $y'' + 9y = 0$. (2pt)

$r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$a = 0,\ b = 3$

$y_g = C_1 e^{at} \cos bt + C_2 e^{at} \sin bt$

$= \boxed{C_1 \cos 3t + C_2 \sin 3t}$

(2) Find all the solutions to the initial value problem $y(\pi/6) = 0, \quad y'(0) = 0$. (2pt)

$y\left(\frac{\pi}{6}\right) = 0 \Rightarrow \cancel{C_1 \cos\left(\frac{\pi}{2}\right)} + C_2 \sin\left(\frac{\pi}{2}\right) = 0$

$\Leftrightarrow C_2 = 0$

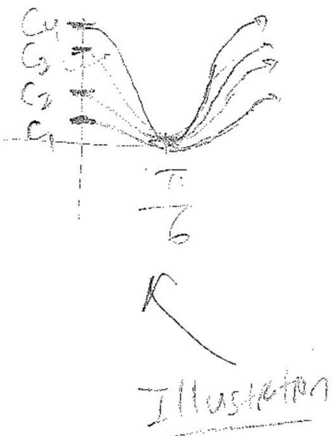
$y'(0) = 0 \Rightarrow \cancel{-3C_1 \sin(3\cdot 0)} + 3C_2 \cos(3\cdot 0) = 0$

$3C_2 = 0$

$\Rightarrow C_2 = 0$

$\boxed{C_2 = 0,\ C_1 \in \mathbb{R}}$

(3) Why does the previous part not violate the 2. order existence and uniqueness theorem? (2pt)



the initial values given for y' and y are at different t locations. different solns. can have slope 0 at $t=0$ and value 0 at $t=\frac{\pi}{6}$, as long as the solution lines don't cross (which they won't)

(4) Find a **particular** solution of

$$y'' + 9y = -9t^2 + 16 \quad (2pt)$$

$y_p = at^2 + bt + c$

$y_p' = 2at + b$

$y_p'' = 2a$

$\Rightarrow y'' + 9y = 2a + 9at^2 + 9bt + 9c = -9t^2 + 16$

$\Rightarrow a = -1,\ b = 0,\ 2a + 9c = 16$

$\Rightarrow c = 2$

$$\boxed{y_p = -t^2 + 2}$$

(5) Find a **particular** solution of

$$y'' + 9y = 12\cos(-3t) \quad (4pt)$$

$\longrightarrow \quad y'' + 9y = 12\cos(3t) \quad \left(\cos(a) = \cos(-a)\right)$

$y_p = A\sin 3t + B\cos 3t$

$y_p' = 3A\cos 3t - 3B\sin 3t$

$y_p'' = -9A\sin 3t - 9B\cos 3t$

$y'' + 9y = -9A\sin 3t - 9B\cos 3t$
$+ 9A\sin 3t + 9B\cos 3t$
$= 0$

try $y_p = At\sin 3t + Bt\cos 3t$

$y_p' = A\sin 3t + 3At\cos 3t + B\cos 3t - 3Bt\sin 3t$

$y_p'' = 3A\cos 3t + 3A\cos 3t - 9At\sin 3t - 3B\sin 3t$
$- 3B\sin 3t - 9Bt\cos 3t$

$= 6A\cos 3t - 6B\sin 3t - 9At\sin 3t - 9Bt\cos 3t$

$y_p'' + 9y_p = 6A\cos 3t - 6B\sin 3t - \cancel{9At\sin 3t} - \cancel{9Bt\cos 3t} + \cancel{9At\sin 3t}$
$\cancel{+ 9Bt\cos 3t}$

$= 12\cos(3t)$

$\Rightarrow B = 0,\ A = 2$

$\boxed{y_p = 2t\sin 3t}$

(6) Give the **general** solution to the following differential equation (Hint: Use Part (1),(4),(5) of this exercise)

$$y'' + 9y = 18t^2 - 32 - 24\cos(-3t) \ (1pt)$$

$= -2(-9t^2 + 16) - 2(12\cos(-3t))$

$= -2(-9t^2 + 16) - 2(12\cos(3t))$

$\leftarrow \cos(a) = \cos(-a)$

$y_g = y_h + y_p$

$= C_1\cos 3t + C_2\sin 3t - 2(-t^2 + 2) - 2(2t\sin 3t)$

$= \boxed{C_1\cos 3t + C_2\sin 3t + 2t^2 - 4 - 4t\sin 3t}$

Exercise 3. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} 2 & -3 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 2, -1$

$\underline{\lambda = 2}: A - \lambda I = \begin{pmatrix} 0 & -3 & 1 & 4 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$(A-\lambda I)\eta_1 = 0 \Rightarrow \eta_1 = \begin{pmatrix} k_1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

let $k_1 = -3$ $\quad k \in \mathbb{R}$

$(A-\lambda I)\eta_2 = \eta_1 \Rightarrow \eta_2 = \begin{pmatrix} k_2 \\ 1 \\ 4 \\ -1 \end{pmatrix}$

$k \in \mathbb{R}$

Solutions: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ -1 \end{pmatrix}$

1^{st} $\quad 1^{st}$ $\quad 2^{nd}$

3 lin. ind. solns.

$\underline{\lambda = -1}: A - \lambda I = \begin{pmatrix} 3 & -3 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$(A-\lambda I)\eta = 0 \Rightarrow \eta = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\boxed{\begin{aligned} y_g = {} & C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} -3 \\ 1 \\ 3 \\ 0 \end{pmatrix} \\ & + C_4 e^{2t} \left[\begin{pmatrix} 1 \\ 1 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right] \end{aligned}}$$

(you may use the next page for ~~additional computations~~)

GENERAL SOLN.

IVP:

$$\begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 - 3c_3 + c_4 \\ c_1 + c_3 + c_4 \\ 8c_3 + 4c_4 \\ -c_4 \end{pmatrix}$$

$\Rightarrow c_4 = 0, \quad c_3 = 0, \quad c_1 = 2,$

$c_2 = 2$

solution to IVP:

$$\boxed{y = 2e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

Exercise 4. (3pt) Consider the differential equation

$$\frac{y''}{e^x} - y' + y = 1.$$

(1) Show that $y_1(x) = e^x$ and $y_2(x) = 1$ are two solutions. (2pt)

$y_1 = e^x$ $\quad$ $y_2 = 1$

$y_1' = e^x$ $\quad$ $y_2' = 0$

$y_1'' = e^x$ $\quad$ $y_2'' = 0$

$\frac{y_1''}{e^x} - y_1' + y_1 = \frac{e^x}{e^x} - \cancel{e^x + e^x} = 1$ ✓ $\quad$ $\frac{y_2''}{e^x} - y_2' + y_2 = \frac{0}{e^x} - 0 + 1 = 1$ ✓

(2) Using (1), find a solution to the associated homogenous equation. (1pt)

$y_h = y_1 - y_2$

$= \boxed{e^x - 1}$

Exercise 5. (9pt) Let

$$A = \begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (6pt)

$4i$

$T = 4+2 = 6 \quad | \quad D = 8+5 = 13$

$\lambda^2 - 6\lambda + 13 = 0 \qquad \lambda = \dfrac{6 \pm \sqrt{36-52}}{2} = 3 \pm 2i$

take $\lambda = 3+2i$

$(A - I\lambda)\eta = \begin{pmatrix} 1-2i & 5 \\ -1 & -1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad -v_1 + v_2(-1-2i) = 0$

$v_2 = -1, \ v_1 = 1+2i$

$\Rightarrow \eta = \begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$

let $z = e^{(3+2i)t}\begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$

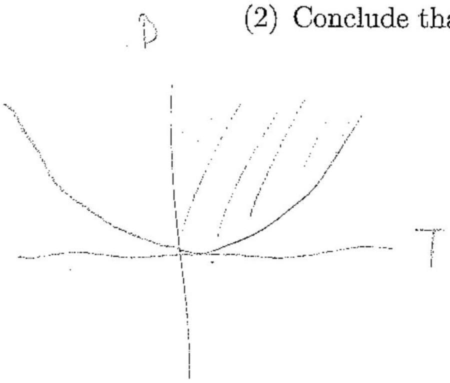
$= e^{3t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1+2i \\ -1 \end{pmatrix}$

$= e^{3t}\begin{pmatrix} \cos 2t + 2i\cos 2t + i\sin 2t - 2\sin 2t \\ -\cos 2t - i\sin 2t \end{pmatrix}$

$y_g = C_1 \operatorname{Re}(z) + C_2 \operatorname{Im}(z)$

$= \boxed{C_1 e^{3t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix}}$

(2) Conclude that the equilibrium point is a spiral. (1pt)



since $T^2 - 4D$

$= 36 - 52 = -16 < 0$

AND $T \neq 0$, $(T = 6)$

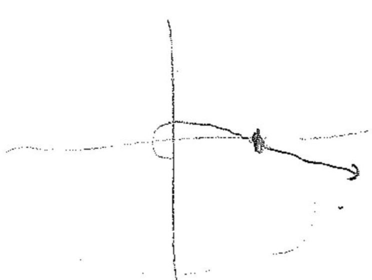
the equilibrium point is a spiral.

(3) Is it a sink or a source? (1pt)

Since $T > 0$ & what I said above, it is a $\boxed{\text{source}}$

(4) Does the spiral rotate clockwise or counterclockwise? (1pt)

$$\begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$



← $\boxed{\text{clockwise}}$

Exercise 6. (8pt)
~~Consider the~~ differential equation

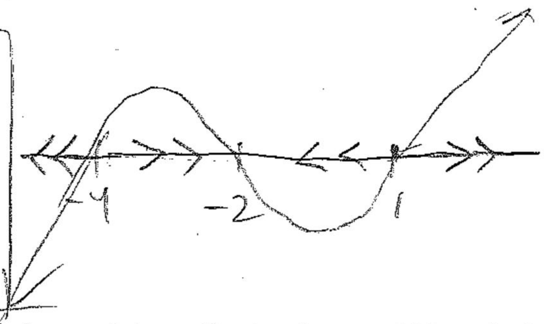
$$\frac{dx}{dt} = e^{-x}(x^3 + 5x^2 + 2x - 8)$$

>0

(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

eq. pts:

$$\boxed{\begin{aligned} x &= 1 \\ x &= -2 \\ x &= -4 \end{aligned}}$$

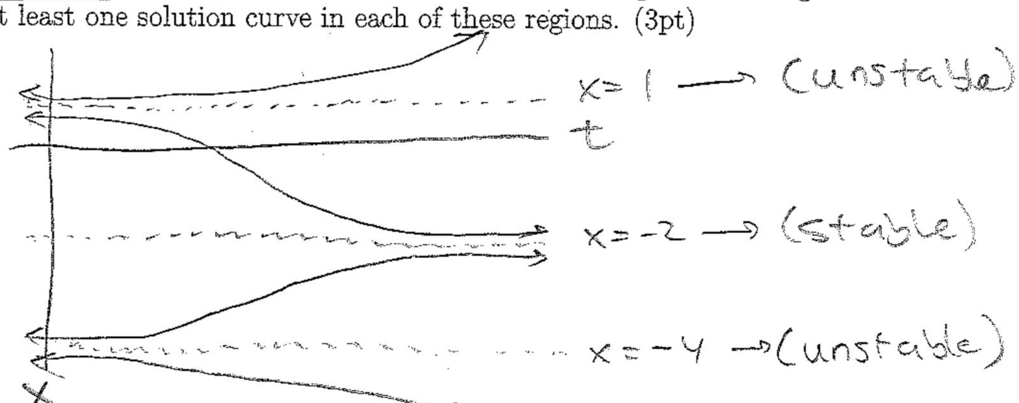


$$\begin{array}{c|cccc} 1 & 1 & 5 & 2 & -8 \\ & & 1 & 6 & 8 \\ \hline & 1 & 6 & 8 & 0 \end{array}$$

$(x-1)(x^2+6x+8)$

$(x-1)(x+2)(x+4)$

(2) Sketch the equilibrium points on the tx-plane and identify the stable ~~and unstable~~ points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)



(3) Does there exist a solution $x(t)$ of the ~~equation~~ satisfying $x(0) = 1$ and $x(1) = 0$? (1pt) Justify your answer. (1pt)

$\boxed{\text{NO}}$

$\frac{dx}{dt}$ is cont. & def on the plane,

$$\frac{d\left(\frac{dx}{dt}\right)}{dx} = e^{-x}(3x^2 + 10x + 2) + (-e^{-x})(x^3 + 5x^2 + 2x - 8)$$

is cont & def on the plane.

Existance & Uniqueness apply.

Only 1 solution exists that crosses $x(0) = 1$, which we know is the equilibrium solution $x = 1$.

if $x(0) = 1$, $x(t)$ must also equal 1.

$\Rightarrow x(1) \neq 0 \Rightarrow$ not possible

$-x^2+3x-(-2x^2+4x+3x-6)$

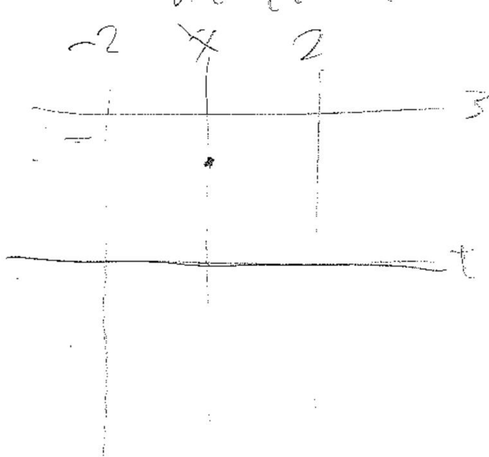
$= x^2-4x+6$

Exercise 7. (5pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x-2}{(t^2-4)(-x^2+3x)}$$

(1) Consider the initial value problem $x_0(0) = 2$. Find the biggest rectangle around this point for which you can apply the existence and the uniqueness theorem. (3pt)

$\frac{dx}{dt} = -\frac{x-2}{(t+2)(t-2)(x)(x-3)} \Rightarrow x \neq 0, 3 \quad t \neq 2, -2$



$\frac{d\left(\frac{dx}{dt}\right)}{dx} = \frac{1}{t^2-4} \frac{(-x^2+3x)-(x-2)(-2x+3)}{(-x^2-3x)^2}$

$= \frac{1}{t^2-4} \frac{x^2-4x+6}{(x^2-3x)^2}$

both $\frac{dx}{dt}$ and $\frac{d\left(\frac{dx}{dt}\right)}{dx}$ exist & are cont.

Rect:

$t \in (-2, 2)$

$x \in (0, 3)$

(2) Can $x_0(1) = 1$? (1pt) Justify your answer. (1pt)

NO $\left.\frac{dx}{dt}\right|_{x=2} = 0$

$\Rightarrow x=2$ is an equilibrium solution.

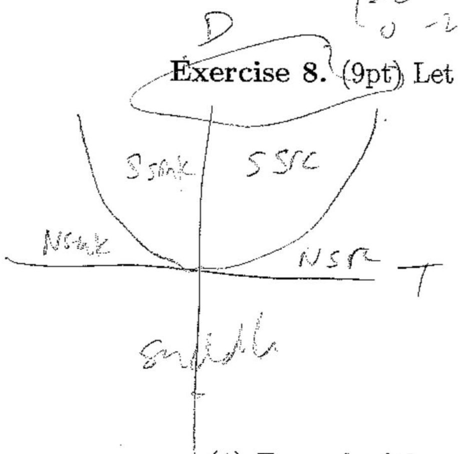
if $x_0(0) = 2$, $x_0(t) = 2$ as well,

so $x_0(1) \neq 1$

$t^2 - 2t -$
$t = 1$
$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$\lambda^2 - 7\lambda + 10 = 0$
$(\lambda-5)(\lambda-2)$
$\begin{pmatrix} -4 & -1 \\ 4 & 1 \end{pmatrix}$ $v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
$\begin{pmatrix} -1 & -1 \\ 4 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercise 8. (9pt) Let



$$A = \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix} \quad T = 7, \quad D = 10$$
$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad T = -2, \quad D = 1$$
$$C = \begin{pmatrix} 2 & 1 \\ -4 & 6 \end{pmatrix} \quad T = 8, \quad D = 16$$
$$D = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \quad T = 0 \quad D = -4$$

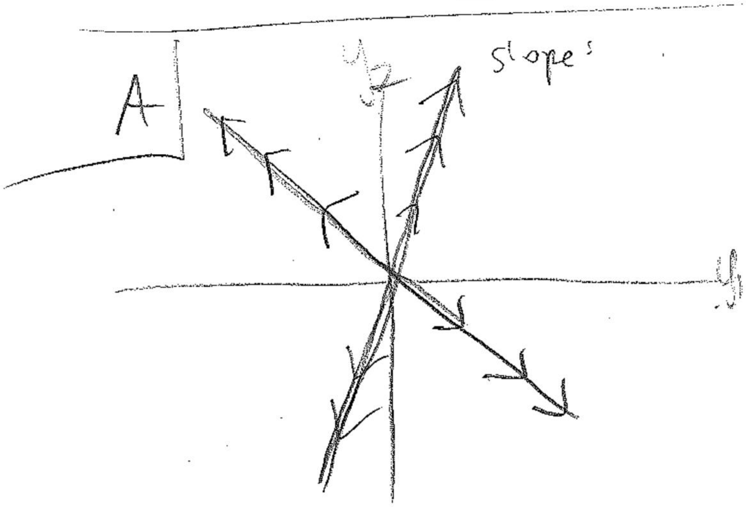
(1) For each of the matrices above decide wether they are a saddle, nodal sink/source, degenerated nodal sink/source, star (sink/source) (7pt)
(2) For A and B draw a **rough** sketch of the half line solutions. (You do **NOT** need to determine the actual eigenvectors!!!!)(2pt)

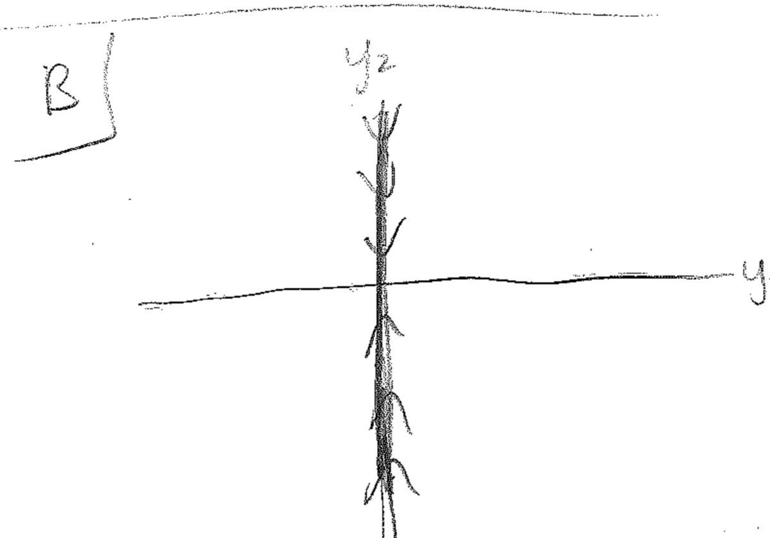
A: $T^2 - 4D$
$= 49 - 40 = 9 > 0$
$T > 0$, $D > 0$ $\Rightarrow$ nodal source

B: $T^2 - 4D = 4 - 4 = 0$
$T < 0$
$\Rightarrow$ degenerated nodal sink

C: $T^2 - 4D$
$= 64 - 64 = 0$
$T > 0$, $D > 0$ $\Rightarrow$ degenerated nodal source

D: $T^2 - 4D = 16 > 0$
$T = 0$, $D < 0$
saddle (symmetrical)





Exercise 9. (3pt)
Consider the third order equation $y''' + 3\cos(2t)y'' - 2y' - (t^2+1)y = -3e^t$. Write this equations as a planar system of first-order equations.

$(y_1 = y')$

$(y_2 = y'')$

$(y_3 = y''')$

$$y_1 = y'$$

$$y_2 = y_1'$$

$$y_3 = -3e^t - 3\cos(2t)\,y_2 + 2y_1 + (t^2+1)\,y$$

Exercise 10. (8pt)
A roast is removed from the oven when its temperature is 270, and 10 minutes later its temperature is 170. Assume Newton's law of cooling, and a room temperature is 70. How much (after removing it from the oven) should you allow the roast to cool, if you want to serve it when its temperature is 150? Since calculators are not allowed, leave your answer in logarithms. Remember that Newton's law of cooling says $dT/dt = k(T - A)$, where T is temperature, t is time, A is surrounding temperature and k is a constant.

$T(0) = 270$

$T(10) = 170$

$A = 70$

$T(\tau) = 150, \ \tau = ?$

$$\int_{T_0}^{T} \frac{dT}{T-A} = \int_0^t k\,dt$$

$$\ln\left(\frac{T-A}{T_0-A}\right) = kt$$

$$T - A = e^{kt}(T_0 - A)$$

$$T = A + (T_0 - A)e^{kt}$$

$$T(10) = A + (T_0 - A)e^{10k}$$

$$170 = 70 + (270-70)e^{10k} = 70 + 200e^{10k}$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{10k} \Rightarrow -\ln 2 = 10k \Rightarrow k = -\frac{\ln 2}{10}$$

$$T(\tau) = 150 = A + (T_0 - A)e^{k\tau}$$

$$150 = 70 + (270-70)e^{\left(-\frac{\ln 2}{10}\right)\tau}$$

$$\frac{80}{200} = \frac{2}{5} = e^{\left(-\frac{\ln 2}{10}\right)\tau}$$

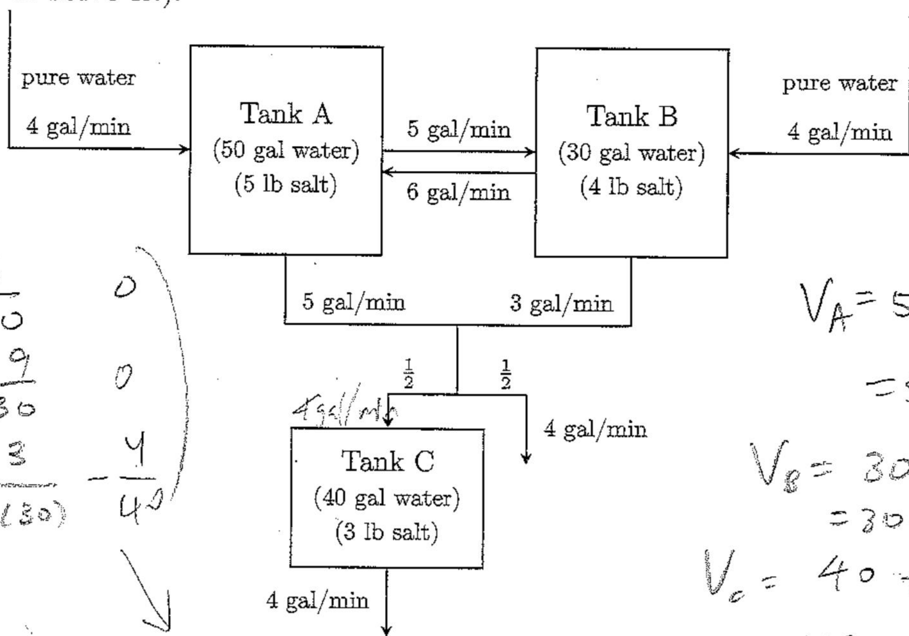
$$\ln\left(\frac{2}{5}\right) = -\frac{\ln 2}{10}\tau$$

(you may use the next page for additional computations)

$$\tau = \frac{10(\ln 5 - \ln 2)}{\ln 2} = \boxed{10\left(\frac{\ln 5}{\ln 2} - 1\right)}$$

minutes

Exercise 11. (9pt) Set up an **initial value problem** that models the salt content ~~$x_A(t)$ and~~ $x_B(t)$ and $x_C(t)$ (in lb) in tank A, B, and C at time t (**you do NOT have to solve it!**).



$$\begin{pmatrix} -\frac{10}{50} & \frac{6}{30} & 0 \\ \frac{5}{50} & -\frac{9}{30} & 0 \\ \frac{5}{2(50)} & \frac{3}{2(30)} & -\frac{4}{40} \end{pmatrix}$$

$V_A = 50 + \cancel{4t} - \cancel{5t} + \cancel{6t} - \cancel{5t}$

$= 50$ gal

$V_B = 30 + \cancel{4t} - \cancel{6t} - \cancel{3t} + \cancel{5t}$

$= 30$ gal

$V_C = 40 + \cancel{4t} - \cancel{4t}$

$= 40$ gal

Please write your final solution below (in particular write it as a matrix equation):

$$\begin{pmatrix} x'_A \\ x'_B \\ x'_C \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{10} & -\frac{3}{10} & 0 \\ \frac{1}{20} & \frac{1}{20} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \quad \begin{pmatrix} x_A(0) \\ x_B(0) \\ x_C(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

(*you may use the next page for additional computations*)

$$\left(e^{y^b}+2y^2e^{y^b}\right)(a+1)x^a.$$

Exercise 12. (4pt) Consider the following differential equation:

$$4y\,\mathrm{dx} + x(1+2y^2)\,\mathrm{dy}$$

Find **integers** a and b such that $x^a e^{y^b}$ is an integration factor of the above equation.

$$4yx^a e^{y^b}\,dx + x(1+2y^2)x^a e^{y^b}\,dy = 0$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$ye^{y^b} \Rightarrow yby^{b-1}e^{y^b}$$

$$4x^a\left(e^{y^b}+by^be^{y^b}\right) = (1+2y^2)e^{y^b}(a+1)x^a$$

$$4x^a e^{y^b} + 4bx^a y^b e^{y^b} = (a+1)x^a e^{y^b} + 2(a+1)x^a y^2 e^{y^b}$$

match terms

$$\Rightarrow 4\cancel{x^a}\cancel{e^{y^b}} = (a+1)\cancel{x^a}e^{y^b}$$

$$\Rightarrow 4b\cancel{x^a}y^b\cancel{e^{y^b}} = 2(a+1)\cancel{x^a}y^2\cancel{e^{y^b}}$$

$$\Rightarrow a+1=4, \quad b=2$$

$$\boxed{a=3,\ b=2}$$

Exercise 13. ($\cancel{12}$pt) Consider the following differential equation:

$$y'' - \frac{6}{t}y' + \frac{10}{t^2}y = 12t^4$$

(1) Decide which of following functions form a fundamental set of solutions to the associated homogeneous equation: (6pt)

$$t^2, t^3, t^4, t^5, t^6$$

$$y'' - \frac{6}{t}y' + \frac{10}{t^2}y = 0$$

$+10$	t^2	t^3	t^4	t^5	t^6
-6	$2t$	$3t^2$	$4t^3$	$5t^4$	$6t^5$
1	2	$6t$	$12t^2$	$20t^3$	$30t^4$
	✓	X	X	✓	X

$2 - \frac{6}{t}2t + \frac{10}{t^2}t^2$

$2 - 12 + 10 = 0$

$20t^3 - \frac{6}{t}5(t^4) + \frac{10}{t^2}t^5$

$20 - 30 - 30 = 0$ ✓

$$\boxed{\begin{aligned} y_1 &= t^2 \\ y_2 &= t^5 \end{aligned}}$$

$W = y_1 y_2' - y_1' y_2$

$= t^2(5t^4) - 2t(t^5)$

$= 3t^6 \neq 0$

$\Rightarrow$ lin ind.

answer

(2) Using variation of parameter, find the **general** solution to the inhomogeneous linear 2.order differential equation. (6pt)

$y_p = v_1 y_1 + v_2 y_2$, $y_1 = t^2$, $y_2 = t^5$

$W = t^2(5t^4) - 2t(t^5)$

$= 3t^6$

$g = 12t^4$

$$v_1 = \int \frac{g y_1}{W} dt$$

$$= \int \frac{12\cancel{t^4}\,\cancel{t^2}}{3\cancel{t^6}} dt$$

$$= \int 4\,dt = 4t$$

$$v_2 = \int \frac{-g y_2}{W} dt$$

$$= \int \frac{-12t^4 t^5}{3t^6} dt$$

$$= \int -4t^3\,dt$$

$$= -t^4$$

$y_p = v_1 y_1 + v_2 y_2$

$= t^2(4t) + (-t^4)t^5$

$= 4t^3 - t^9$

$y_g = y_h + y_p$,

$y_h = C_1 y_1 + C_2 y_2$

$y_g =$ $\boxed{C_1 t^2 + C_2 t^5 + 4t^3 - t^9}$

Extra page

Extra page

Extra page