134

FINAL

12/10/2018

Name: Hyounjun Chang

section: 2C

Math33B
Nadja Hempel
nadja@math.ucla.edu

UID: 105-132-882

Problem	Points	Score
1	7	
2	5	
3	10	
4	10	
5	8	
6	9	
7	9	
8	8	
9	9	
10	9	
11	11	
12	5	
Total	100	

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Solve the following equation. (*Hint: Find the integrating factor*)

$$\overset{P}{(x^2+y^2)}\,dx - \overset{Q}{2xy}\,dy = 0$$

$\frac{\partial P}{\partial y} = 2y$ $\quad \frac{\partial Q}{\partial x} = -2y$ $\quad$ not exact

$\frac{d}{dy}(\mu P) = \frac{d}{dx}(\mu Q)$ $\quad$ let u be funct. of x then...

$\frac{du}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)u = \frac{1}{-2xy}(2y+2y) = \frac{4y}{-2xy} = -2x\,u$

$u = e^{\int -2x\,dx} = e^{-x^2}$

$e^{-x^2}(x^2+y^2)dx - e^{-x^2}(2xy)\,dy = 0$

$= (x^2e^{-x^2} + e^{-x^2}y^2)\,dx - 2xye^{-x^2}\,dy = 0$

$F(x,y) = xe^{-x^2}y^2 + x(\phi) = C$

$\frac{dF}{dx} = e^{-x^2}y^2 + -2x^2e^{-x^2}y^2 + x'(\phi) = \mu P = e^{-x^2}y^2 + e^{-x^2}x^2$

$x'(\phi) = e^{-x^2}x^2 + 2x^2e^{-x^2}y^2$

$x'(\phi) = (1+2x^2y^2)e^{-x^2}$

$F(x,y) = xe^{-x^2}y^2 + \int (1+2x^2y^2)e^{-x^2}\,dx = C$

Exercise 2. (5pt) Solve $y' = y(y+1)(x+2)(x+3)$

$$\frac{dy}{dx} = y\,(y+1)(x+2)(x+3)$$

$$\int \frac{1}{y(y+1)}\,dy = \int (x+2)(x+3)\,dx \quad \leftarrow x^2+5x+6$$

$$\frac{A}{y} + \frac{B}{y+1} = \frac{1}{y(y+1)} \qquad A(y+1) + B(y) = 1$$

let $y=0$ $A=1$ $y=-1$ $B=-1$

$$= \int \frac{1}{y}\,dy + \int \frac{-1}{y+1} = \int x^2+5x+6\,dx$$

$$\ln|y| - \ln|y+1| = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C$$

$$\frac{y}{y+1} = \pm\left(e^C \; e^{\frac{1}{3}x^3+\frac{5}{2}x^2+6x}\right)$$

Exercise 3. (10pt) Find a **particular** solution to the following two differential equations

(1) $y'' + 4y = 8t^2 - 4t$ (2pt)

Let $y = a_2 t^2 + a_1 t + a_0$

$y' = 2a_2 t + a_1$

$y'' = 2a_2$

$2a_2 + 4a_2 t^2 + 4a_1 t + a_0 = 8t^2 - 4t$

$2a_2 + a_0 = 0$ $\quad 4a_2 = 8$ $\quad a_2 = \underline{2}$ $\quad a_0 = \underline{-4}$

$4a_1 = -4$ $\quad \underline{a_1 = -1}$ $\quad \boxed{y = 2t^2 - t - 4}$

(2) $y'' + 4y = 4\sin(2t)$ (4pt)

$y = at\cos(2t) + bt\sin(2t)$

$y' = a\cos 2t - 2at\sin(2t) + b\sin(2t) + 2bt\cos(2t)$

$y'' = -2a\sin 2t - 2a\sin 2t - 4at\cos(2t)$

$+2b\cos 2t + 2b\cos(2t) - 4bt\cos(2t)$

$= -4a\sin 2t - 4at\cos 2t + 4b\cos 2t - 4bt\cos(2t)$

$+ 4at\cos(2t) + 4bt\sin 2t = 4\sin 2t$

$-4a\sin 2t + 4b\cos(2t) = 4\sin(2t)$

$b = 0$ $\quad a = -1$

$\boxed{y = -t\cos(2t)}$

(3) Give the **general** solution to the following differential equation

$$y'' + 4y = 8\sin(2t) - 8t^2 + 4t. \text{ (4pt)}$$

$y_{gen} = y_{part} + y_h$

y_h: $\lambda^2 + 4 = 0$ $\quad \lambda = \pm 2i$ $\quad e^{2ti} = \underbrace{\cos 2t}_{sol\ 1} + \underbrace{i \sin 2t}_{sol\ 2}$

$2 \cdot$

$8\sin(2t) = 2(4\sin(2t))$ $\quad -8t^2 + 4t = -(8t^2 - 4t)$

$\alpha = 2$ $\quad \alpha y_f = -2t\cos 2t$

$2 y_f$

$b = -1$ $\quad b y_g = -(2t^2 - t - 4)$

$-1\, y_g$

$y_p = \alpha y_f + b y_g$

$$\boxed{y_{gen} = (-2t\cos 2t) - (2t^2 - t - 4) + C_1 \cos 2t + C_2 \sin 2t}$$

Exercise 4. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 1 \\ -6 \\ -2 \end{pmatrix}$$

$$p(\lambda) = (-1-\lambda)^3 \, (3-\lambda)$$

$\lambda = -1, \; 3.$

$$\lambda = 3 \quad \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0 \qquad a_4 = 0, \quad -a_2 + a_3 = 0, \quad a_2 = a_3$$

$$-a_1 + 2a_3 = 0 \implies a_1 = 2a_3$$

$$\vec{v}_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1 \quad \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = 0 \qquad a_2 = 0 \;\; a_3 = 0 \;\; a_4 = 0$$

$\dim = \underline{1}$ $\qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\text{try } (A-\lambda I)^2 \quad \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \dim 2 < \text{AlgMult}(-1)$$

$$V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A-\lambda I)^3 \quad \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 8 & -6 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 64 & -48 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rows are lin multiples

$$8a_3 - 6a_4 = 0$$
$$4a_3 - 3a_4 = 0$$
$$a_3 = \tfrac{3}{4} a_4$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 3/4 \\ 1 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_3 = \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (A-\lambda I)\vec{v}_3 = \begin{bmatrix} 0 \\ 3/4 \\ 0 \\ 0 \end{bmatrix} \qquad (A-\lambda I)\vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

General Solution:

$$t\,(A-\lambda I)\,v_3 \qquad \frac{t^2}{2}\,(A-\lambda I)^2 v_3$$

$$c_1 e^{3t}\begin{pmatrix}2\\1\\1\\0\end{pmatrix} + c_2 e^{-t}\left(\begin{pmatrix}0\\0\\3/4\\1\end{pmatrix} + t\begin{pmatrix}0\\3/4\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}3/2\\0\\0\\0\end{pmatrix}\right)$$

$$+ c_3 e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\0\\0\end{pmatrix}\right) + c_4 e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$t\,(A-\lambda I)\,v_3$$

Initial value $y(0) = \begin{pmatrix}3\\1\\-6\\-2\end{pmatrix}$

$$c_1\begin{pmatrix}2\\1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\0\\3/4\\1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + c_4\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}3\\1\\-6\\-2\end{pmatrix}$$

$$\begin{bmatrix}2&0&0&1\\1&0&1&0\\1&3/4&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = \begin{pmatrix}3\\1\\-6\\-2\end{pmatrix}$$

$\underline{c_2 = -2}$ $\qquad c_1 - 3/2 = -\frac{12}{2}$ $\qquad \underline{c_1 = -\frac{9}{2}}$

$-\frac{9}{2} + c_3 = 1 \qquad \underline{c_3 = \frac{11}{2}} \qquad -9 + c_4 = 3$

$\underline{c_4 = 12}$

Sol. to Init val.

$$-\frac{9}{2}e^{3t}\begin{pmatrix}2\\1\\1\\0\end{pmatrix} - 2e^{-t}\left(\begin{pmatrix}0\\0\\3/4\\1\end{pmatrix} + t\begin{pmatrix}0\\3/4\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}3/2\\0\\0\\0\end{pmatrix}\right)$$

$$+ \frac{11}{2}e^{-t}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\0\\0\end{pmatrix}\right) + 12e^{-t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Exercise 5. (8pt) Consider the differential equation

$$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t - 1) - t(e^t+1), \qquad (t>0)$$

(1) Show that $y_1 = e^t(2t+1) - (t+1)$ is solutions to the above equation. (4pt)
(Show ALL your calculations in detail for full credit)

$y_1' = e^t(2t+1) + 2e^t - 1$

$y_1'' = e^t(2t+1) + 2e^t + 2e^t = e^t(2t+1) + 4e^t$

$t^2(e^t(2t+1) + 4e^t) - (t^2+2t)(e^t(2t+1) + 2e^t - 1)$

$+ (t+2)(e^t(2t+1) - (t+1))$

$= t^2e^t(2t+1) + 4t^2e^t - t^2e^t(2t+1) - \cancel{2t^2e^t} + \cancel{t^2}$

$- 2t\, e^t(2t+1) - \cancel{4te^t} + \cancel{2t} + \cancel{te^t(2t+1)} - \cancel{t^2} - \cancel{t}$

$\cancel{4te^t} + 2e^t - \cancel{2t} - 2$

$= \cancel{2t^2e^t} - t\,e^t(2t+1) - t + 2e^t - 2$

$\cancel{-2t^2e^t} - te^t - t + 2e^t - 2$

$= -t(e^t+1) + 2(e^t-1)$

$= 2(e^t-1) - t(e^t+1)$

$te^t + e^t + t - 1 \qquad -te^t + e^t + 2t - 1$

(2) Given that $y_2 = e^t(t+1)+(t-1)$, and $y_3 = e^t(1-t)+(2t-1)$ are also solutions to the above equation, find the general solution to the equation. Justify your answer. (4pt)

Wronskian: $y_2 y_3' - y_2' y_3$

$y_2' = e^t(t+1) + e^t + 1 = te^t + 2e^t + 1$

$y_3' = e^t(1-t) - e^t + 2 = -te^t + 2$

$W_{y_2,y_3} = \left(e^t(t+1) + (t-1)\right)\left(-te^t+2\right) - \left(te^t+2e^t+1\right)\left(e^t(1-t)+(2t-1)\right)$

$\frac{y_2}{y_3} = \frac{te^t+e^t+t-1}{-te^t+e^t+2t-1} \neq C \leftarrow$ long division

So: y_2 and y_3 are linearly independent

gen sol. = $\boxed{y(t) = C_1 y_2 + C_2 y_3}$

orig problem is of 2nd order, 2 linear combinations of independent solutions solve the problem.

Exercise 6. (9pt) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (5pt)

Let $\vec{y} = e^{\lambda t}\vec{v}$

$p(\lambda) = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 3 + 2 = \lambda^2 - 4\lambda + 5$

$\lambda = \dfrac{4 \pm \sqrt{16-20}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$

$\lambda = 2+i \quad \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

If $x = 1 \quad (-1-i) = 2y \quad y = -\frac{1}{2} - \frac{i}{2} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{i}{2} \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$

$e^{(2+i)t} v = e^{2t} e^{it} v = e^{2t}(\cos t + i \sin t)\left(\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right)$

$= e^{2t}\left(\cos t \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right) + i\, e^{2t}\left(\cos t \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\right)$

Gen sol:

$$\boxed{c_1 e^{2t}\left(\cos t \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right) + c_2 e^{2t}\left(\cos t \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\right)}$$

(2) Conclude that the equilibrium point is a spiral. (1pt)

both λ is imaginary and real component of $\lambda \neq 0$

So it's spiral (there's sine and cosines)

$Re(\lambda) \neq 0 \quad T^2 - 4D < 0$

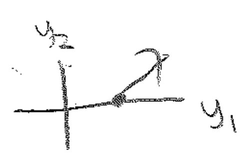
(3) Is it a sink or a source? (1pt)

$\boxed{\text{Sorce}}$

$\lim_{t \to \infty} y(t) = \infty$ goes away

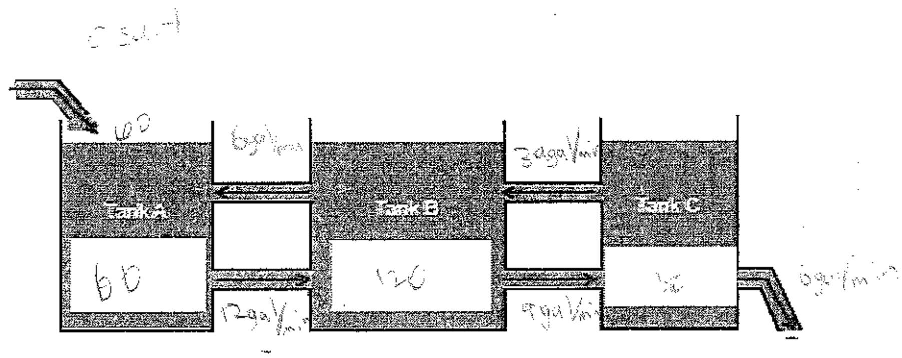
(4) Does the spiral rotate clockwise or counterclockwise? (2pt)

$y' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$\boxed{\text{canterclockwise}}$

Prob diag 2018.pdf



Exercise 7. (9pt)
Consider the above mixing problem with the following data.

- at time $t = 0$ there is 0 lb of salt in tank A, 10 lb of salt in tank B, and 20 lb of salt in tank C.
- Tank A contains initially 60 gallons of solution, Tank B contains initially 120 gallons of solution and Tank C contains initially 30 gallons of solution.
- 6 gal/min of water enters tank A through the top far left pipe.
- The solutions flows at
  - at 6 gal/min through the upper left pipe
  - at 12 gal/min through the lower left pipe
  - at 3 gal/min through the upper right pipe
  - at 9 gal/min through the lower right pipe
- 6 gal/min of solutions leaves tank C through the bottom far right pipe.

Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ in tank A, B, and C at time t (**you do NOT have to solve it!**).

$x_A(0)=0 \quad x_B(0)=10 \quad x_C(0)=20$

$V_A = 60 + 6t + 6t - 12t = 60$

Volume

$V_B = 120 + 3t - 6t + 12t - 9t = 120$

const. volume

$V_C = 30 + 9t - 3t - 6t = 30$.

$x_A' = -\frac{12}{60} x_A + \frac{6}{120} x_B$

$x_B' = \frac{12}{60} x_A - \frac{15}{120} x_B + \frac{3}{30} x_C$

$x_C' = 0 + \frac{9}{120} x_B - \frac{9}{30} x_C$

This is same as

$$\boxed{\vec{y}\,' = \begin{pmatrix} -\frac{1}{5} & \frac{1}{20} & 0 \\ \frac{1}{5} & -\frac{15}{120} & \frac{1}{10} \\ 0 & \frac{9}{120} & -\frac{3}{10} \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \qquad x_A(0)=0 \quad x_B(0)=10 \quad x_C(0)=20}$$

$\vec{y}\,' = A y$

$\frac{15}{27}\ \frac{65}{27}\ -\frac{81}{27}$

$\frac{1}{27}-\frac{5}{9}+\frac{7}{3}-3$

$$\begin{array}{r} x^2+2x+1 \\ (x-3)\overline{)\,x^3-5x^2+7x-3} \\ x^3-3x^2 \\ \hline -2x^2+7x \\ -2x^2+6x \\ \hline x-3 \end{array}$$

Exercise 8. (8pt)
Consider the differential equation

$$\frac{dx}{dt} = e^x(x^3 - 5x^2 + 7x - 3)$$

(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

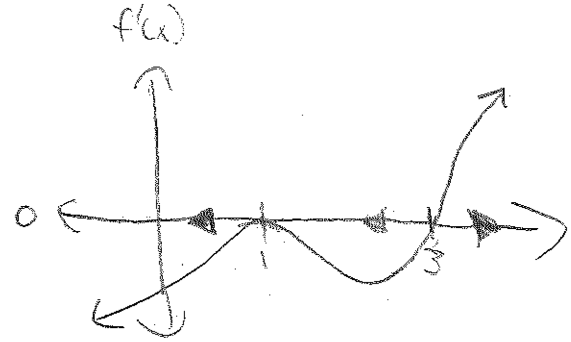
$\frac{dx}{dt} = e^x(x-3)(x-1)^2$

equilibrium points: $x=3$, $x=1$

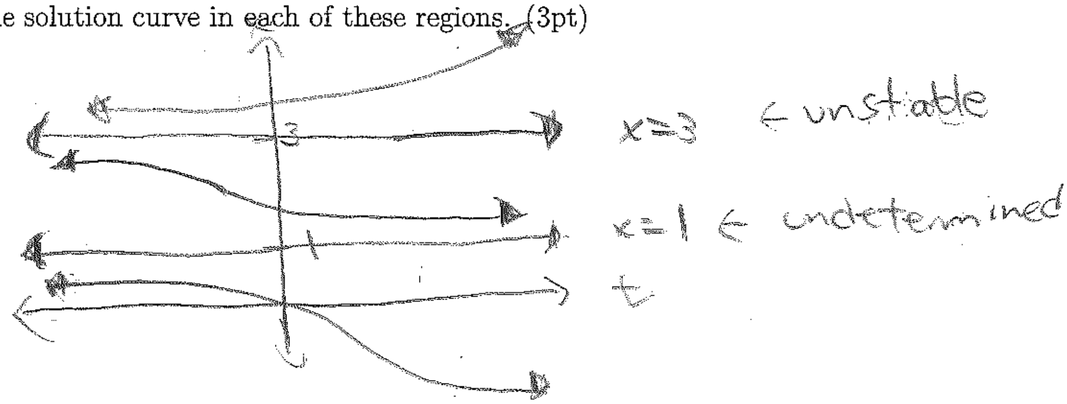
$x>3$ $(+)(+)(+)$

$1<x<3$ $+\ -\ +\ =-$

$+\ -\ +\ =-$



(2) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)



(3) Does there exist a solution of the equation, $x(t)$, satisfying $x(0) = -1$ and $x(2) = 0$? Justify your answer. (2pt)

No. when $x<1$, $\frac{dx}{dt}$ is negative.

so it must be decreasing.

$0>-1$ and since $x(0)<x(2)$, this

can not happen.

Exercise 9. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx} \qquad \frac{(x-2)(x-1)}{tx}$$

(1) Find all constant solutions of the above equation. (4pt)

$\frac{dx}{dt} = 0$ when $x = 2, 1$

$\boxed{x = 2, \quad x = 1}$ are const. sol.

(2) Argue that the range of the solution to the initial value problem $x(1) = 1.2$ is contained in $(1, 2)$. (3pt)

$\frac{dx}{dt}$ at $(1,2)$ is neg $\frac{-}{+}$

It would be less than 1.2, but greater than $x(2)$

(3) Can you apply the existence theorem to the initial value problem $\cancel{y}(0) = 5$? (1pt) Justify your answer. (1pt)

No you can't

there is no defined derivative for $t=0$

(denominator $= 0$)

Exercise 10. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$\overset{P}{2(x + xy^2)} + \overset{Q}{b(x^m y + y^2)}\frac{dy}{dx} = 0$$

so $\frac{dP}{dy} = \frac{\partial Q}{dx}$

$2(2xy) = b(m x^{m-1} y)$

$m-1 = 1$ $\quad$ $\boxed{m=2}$

$bm = 4$ $\quad$ $\boxed{b=2}$

(2) Solve the equation using the value of b and m you obtained in part (a). (6pt)

$2(x+xy^2)\,dx + 2(x^2y + y^2) = 0$

$(x+xy^2)\,dx + (x^2y+y^2)\,dy = 0$

Integrate w.r.t x = $F = \frac{1}{2}x^2 + \frac{1}{2}x^2y^2 + y(\phi) = C$

$\frac{dF}{dy} = x^2y + y'(\phi) = Q = x^2y + y^2$

$y'(\phi) = y^2 \quad y(\phi) = \frac{1}{3}y^3$

$$\boxed{F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}x^2y^2 + \frac{1}{3}y^3 = C}$$

Exercise 11. (11pt) Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$6^2 - 4(9)$

(1) Determine where in the trace-determinante plane the system $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$ fit. (3pt)

A: $\lambda = 3$, real T: positive D: positive $T^2 - 4D = 0$

Nodal source

B: $\lambda = 2$, real, T: positive, D: positive $T^2 - 4D = 0$

Nodal source

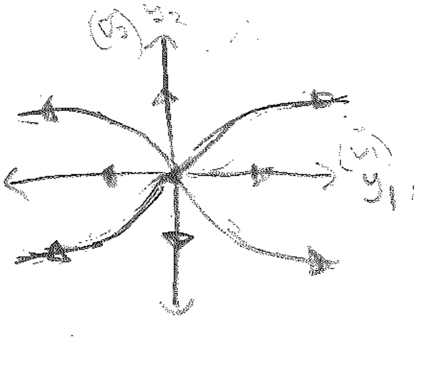
(2) Find all of the half line solutions for the system $\vec{y}' = A\vec{y}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

Finding Eigen vectors for A:

$\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e^{3t}\vec{v_1}$ $\to \lim_{t\to\infty} = \infty$

$\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$ $e^{3t}(v_2 + t\vec{v_1})$ $\lim_{t\to\infty} = \infty$

gen sol: $e^{3t}\left((c_1 + tc_2)\vec{v_1} + c_2\vec{v_2}\right)$



$\lim_{t\to\infty}$ $\downarrow$ $= c_2\vec{v_1}$ direction in

$\lim_{t\to-\infty}$ $\to -c_2\vec{v_1}$ direction

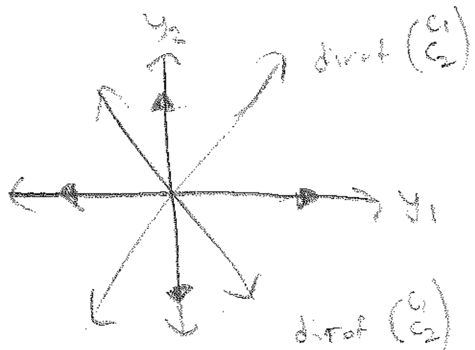
as $t\to\infty$, Solution parallels $c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

as $t\to-\infty$ " $-c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(3) Find all of the half line solutions for the system $\vec{x}' = B\vec{x}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$\lambda = 2$ finding eigenvectors $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

gen sol: $C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{2t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$



$\lim_{t \to \infty} y_1(t) = \infty$

$\lim_{t \to \infty} y_2(t) = \infty$

Exercise 12. (5pt)

(1) Consider the second order equation $y'' + 3t^2y' - \cos(t)y = -3e^t$. Write this equations as a planar system of first-order equations. (2pt)

$y'' = -3t^2y' + \cos(t)y - 3e^t$

let $x = y'$ then $x' = y''$

$$\boxed{x' = -3t^2x + \cos(t)y - 3e^t}$$

(2) Consider more generally an n-order equation $y^{(n)} = F(t, y, \ldots, y^{(n-1)})$. How can you write this as a system of first-order equations? (3pt)

Let $y^{(n-1)} = x_{(n-1)}$ then $y^{n} = x_{(n-1)}'$

$y^{(n-2)} = x_{n-2} \ldots y^{n-3} = x_{n-3} \ldots$ etc,

then $y^{(n)} = F(t, y \ldots y^{(n-1)})$ becomes

$$\boxed{x_{(n-1)}' = F(t, y, a_2, a_3 \ldots, a_{n-2}, a_{n-1})}$$

Extra page