33B Final

Sean Yin

TOTAL POINTS

97 / 100

QUESTION 1

1 homogenous eon 7 / 7

+ **2 pts** Homogeneous
+ **1 pts** Substitution
✓ + **3 pts** Single-Variable Integrating Factor
✓ + **2 pts** Making Exact
✓ + **2 pts** Solving
+ **0 pts** No Points

QUESTION 2

2 separable eqn 5 / 5

✓ - **0 pts** Correct
- **1 pts** minor mistake
- **1 pts** need more simplification
- **5 pts** no work
- **3 pts** know it's separable equation, fail to do the partial fraction decomposition
- **2 pts** right hand side is polynomials of x, integration can be calculated directly.
- **3 pts** idea is correct, need calculation

QUESTION 3

forcing term 10 pts

3.1 polynomial 2 / 2

✓ - **0 pts** Correct
- **0.5 pts** b=-1
- **0.5 pts** c=-1
- **0.5 pts** a=2
- **2 pts** wrong

3.2 sin 4 / 4

✓ - **0 pts** Correct
- **1 pts** missing cos in the Setup/t for cos in Setup/wrong second setup
- **1 pts** computational mistake
- **2 pts** missing 2 in the differential
- **4 pts** wrong/no answer
- **2 pts** missing second step
- **2 pts** computational mistake
- **0.5 pts** missing t in answer
- **0.5 pts** missing - in the answer
- **1 pts** didn't finish
- **1 pts** missing 1 step

3.3 general solution 3 / 4

- **0 pts** Correct
- **1 pts** no/wrong characteristic polynomial
✓ - **1 pts** no/wrong roots
- **1 pts** no/wrong homogeneous solution
- **1 pts** wrong final answer
- **4 pts** wrong/no answer
- **0.5 pts** - missing for polynomial
- **0.5 pts** missing t for cos/wrong answer for trig part

QUESTION 4

4 system 10 / 10

✓ - **0 pts** Correct
- **1 pts** Incorrectly identified the eigenvalues or their algebraic multiplicity.
- **2 pts** Incorrectly found the eigenvectors.
- **2 pts** Incorrectly found generalized eigenvectors.
- **2 pts** Incorrect coefficients or powers of t or (A-L I) in general solution
- **1 pts** Wrong vectors in general solution.
- **2 pts** Failed to solve IVP.
- **1 pts** Arithmatic error
- **1 pts** Got an unsolvable system when solving IVP.

QUESTION 5

2nd linear differential equation 8 pts

5.1 verify 4 / 4

✓ - **0 pts** Correct

- **2 pts** incorrect calculation
- **2 pts** not finished
- **4 pts** no work
- **3 pts** some work

5.2 find general solution **2 / 4**

- **0 pts** Correct
- **4 pts** Incorrect calculation of homogeneous. For second order linear differential equation, use y_g=c1*y_h1+c2*y_h2+y_p. y1,y2,y3 can be decomposed in that way, hence we can get y_h1=y1-y2, y_h2 = y2-y3.

✓ - **2 pts** **incorrect calculation of y_h1, y_h2, but idea is correct**

- **3 pts** incorrect calculation of y_h1, y_h2
- **3 pts** some work
- **1 pts** y_g=c1*y_h1+c2*y_h2+y_p
- **1 pts** no calculation detail

QUESTION 6

linear system 9 pts

6.1 find general solution **5 / 5**

✓ - **0 pts** **Correct**

- **3 pts** eigenvector: solve for (A-\lambda*I)v = 0.
- **2 pts** some calculation error\no finished
- **1 pts** final answer incorrect
- **5 pts** no work
- **3 pts** calculation error, incorrect eigenvalue, eigenvector, idea is correct

6.2 spiral? **1 / 1**

✓ - **0 pts** **Correct**

- **1 pts** incorrect

6.3 sink\source? **1 / 1**

✓ - **0 pts** **Correct**

- **1 pts** incorrect

6.4 direction? **2 / 2**

✓ - **0 pts** **Correct**

- **2 pts** wrong
- **1 pts** Somework

QUESTION 7

7 9 / 9

✓ - **0 pts** **Correct**

- **1 pts** one entry wrong
- **2 pts** two entries wrong
- **3 pts** three entries wrong
- **4 pts** 4 entries wrong
- **5 pts** 5 entries wrong
- **7 pts** all entries wrong
- **1 pts** incorrect initial value
- **2 pts** initial value missing
- **9 pts** wrong/ no answer
- **2 pts** not taking concentration

QUESTION 8

8 pts

8.1 3 / 3

✓ + **1 pts** **Correct Roots**

✓ + **2 pts** **Phase Line**

\+ **0 pts** No Points

8.2 3 / 3

✓ + **1 pts** **Curves**

✓ + **1 pts** **1 Stability**

✓ + **1 pts** **3 Stability**

\+ **0 pts** No Points

8.3 2 / 2

✓ + **1 pts** **Correct**

✓ + **1 pts** **Justification**

\+ **0 pts** No Points

QUESTION 9

9 pts

9.1 4 / 4

✓ - **0 pts** **Correct**

- **4 pts** Didn't know to solve x^2-3x+2=0.
- **2 pts** Got the wrong roots.

9.2 3 / 3

✓ - **0 pts** **Correct**

- **1 pts** Didn't mention uniqueness theorem.
- **2 pts** Didn't say that uniqueness means solution cannot cross the solutions x=1 and x=2.

9.3 2 / 2

✓ - **0 pts** **Correct**

- **1 pts** Wrong answer
- **1 pts** Inadaquate justification.

QUESTION 10

9 pts

10.1 3 / 3

✓ **- 0 pts Correct**

- **- 1 pts** Didn't use the definition of exact.
- **- 2 pts** Incorrectly solved for b and m.
- **- 1 pts** Incorrectly solved for one of b or m.

10.2 6 / 6

✓ **- 0 pts Correct**

- **- 1 pts** Minor Calculation error.
- **- 2 pts** Found antiderivative, but not solution (need to set F(x,y)=C).
- **- 2 pts** Did not use the correct algorithm to solve.
- **- 6 pts** Wrong/Blank
- **- 1 pts** Incorrectly solved for g'(y) or h'(x).

QUESTION 11

11 pts

11.1 3 / 3

✓ **+ 1.5 pts Correct for A**

✓ **+ 1.5 pts Correct for B**

+ **0 pts** No Points

11.2 4 / 4

✓ **+ 2 pts Eigenvector**

✓ **+ 2 pts Sketch**

+ **0 pts** No Points

11.3 4 / 4

✓ **+ 2 pts Two Eigenvectors**

✓ **+ 2 pts Star Behavior**

+ **0 pts** No points

QUESTION 12

5 pts

12.1 2 / 2

✓ **- 0 pts Correct**

- **- 0.5 pts** did not solve for v' (correctly)
- **- 2 pts** no answer/ wrong answer
- **- 1 pts** wrong substitution

12.2 3 / 3

✓ **- 0 pts Correct**

- **- 3 pts** wrong/no answer
- **- 2 pts** for trying
- **- 1 pts** missing F(t, y, ... , y_{n-1}) in answer/missing ' in the answer
- **- 1.5 pts** only using one variable
- **- 1 pts** missing equations in answer
- **- 1 pts** using y^{i} in equations
- **- 1 pts** not adding additional equations

FINAL

12/10/2018

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Problem	Points	Score
1	7	
2	5	
3	10	
4	10	
5	8	
6	9	
7	9	
8	8	
9	9	
10	9	
11	11	
12	5	
Total	100	

Testing wheeeeeeeeeee! ...! Testing
Testing Testing Testing
Testing Testing

Instructions

(1) Enter your name, SID number, and discussion section on the top of this page.
(2) If you need **more space**, use the extra page at the end of the exam.
(3) NO Calculators, computers, books or notes of any kind are allowed.
(4) Show your work. Unsupported answers will receive few or no credit.
(5) Good Luck!

Exercise 1. (7pt) Solve the following equation. (*Hint: Find the integrating factor*)

$$(x^2 + y^2)\, dx - 2xy\, dy = 0$$

$u = e^{\wedge \int} \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{-2xy}\left(2y - (-2y)\right) = \frac{4y}{-2xy} \quad u = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$

$\boxed{u = \frac{1}{x^2}} \Rightarrow \quad \frac{(x^2+y^2)}{x^2} dx - \frac{2xy}{x^2} dy = 0$

$\left(1 + \frac{y^2}{x^2}\right) dx \left(- \frac{2y}{x}\right) dy = 0$

$\int P\, dx = \int 1 + \frac{y^2}{x^2} dx = x - \frac{y^2}{x} + \phi(y)$

$\phi'(y) = -\frac{2y}{x} - \frac{d}{dy}\left(x - \frac{y^2}{x}\right) = -\frac{2y}{x} + \frac{2y}{x} = 0$

$\phi = \int 0\, dy = C$

$\boxed{F(x,y) = x - \frac{y^2}{x} = C}$

Exercise 2. (5pt) Solve $y' = y(y+1)(x+2)(x+3)$

$$\frac{dy}{dx} = y(y+1)(x+2)(x+3)$$

$$\int \frac{1}{y(y+1)}\,dy = \int (x+2)(x+3)\,dx$$

$$\frac{Ay+A}{By} = 0y+1$$

$$Ay+By=0y$$

$$A=1$$

$$B=-1$$

$$\int \frac{1}{y} + \frac{-1}{y+1}$$

$$\int \frac{1}{y}dy + \int \frac{-1}{y+1}dy = \int x^2+5x+6\ dx$$

$$\ln|y| - \ln|y+1| = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C$$

$$\ln\left|\frac{y}{y+1}\right| = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C$$

$$\frac{y}{y+1} = e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}$$

$$y = (y+1)\,e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}$$

$$y - ye^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C} = e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}$$

$$y\left(1 - e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}\right) = e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}$$

$$y = \frac{e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}}{1 - e^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + C}} \qquad A = e^C$$

$$\boxed{y = \frac{Ae^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x}}{1 - Ae^{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x}}}$$

Exercise 3. (10pt) Find a **particular** solution to the following two differential equations

(1) $y'' + 4y = 8t^2 - 4t$ (2pt)

$\lambda^2 + 4 = (\lambda+2)(\lambda-2)$

guess $y = At^2 + Bt + C$

$y' = 2At + B$

$y'' = 2A$

$2A + 4At^2 + 4Bt + 4C = 8t^2 - 4t$

$4A = 8$ $\quad A = 2$

$4B = -4$ $\quad B = -1$

$2A + 4C = 0$ $\quad C = -1$

$\boxed{y_p = 2t^2 - t - 1}$

(2) $y'' + 4y = 4\sin(2t)$ (4pt)

guess $y = A\sin 2t + B\cos 2t$

$y' = 2A\cos 2t - 2B\sin 2t$

$y'' = -4A\sin 2t - 4B\cos 2t$

$-4A\sin 2t - 4B\cos 2t + 4A\sin 2t + 4B\cos 2t = 0$

$y = At\sin 2t + Bt\cos 2t$

$y' = A\sin 2t + 2At\cos 2t + B\cos 2t - 2Bt\sin 2t$

$y'' = 2A\cos 2t + 2A\cos 2t - 4At\sin 2t - 2B\sin 2t - 2B\sin 2t - 4Bt\cos 2t$

$4A\cos 2t - 4At\sin 2t - 4B\sin 2t - 4Bt\cos 2t + 4At\sin 2t + 4Bt\cos 2t$

$4A\cos 2t - 4B\sin 2t = 4\sin(2t)$

$-4B = 4$ $\quad B = -1$

$4A = 0$ $\quad A = 0$

$\boxed{y_p = -t\cos(2t)}$

(3) Give the **general** solution to the following differential equation

$$y'' + 4y = 8\sin(2t) - 8t^2 + 4t. \text{ (4pt)}$$

① $4A = -8 \quad A = -2$
$4B = 4 \quad B = 1$
$2A + 4C = 0 \quad C = 1$

$= -2t^2 + t + 1$

② $-4B = 8 \quad B = -2$
$4A = 0 \quad A = 0$

$= -2t\cos(2t)$

③ $(\lambda^2 + 0\lambda + 4) = (\lambda+2)(\lambda-2)$

$\lambda = -2, 2$

$y_h = C_1 e^{-2t} + C_2 e^{2t}$

$y = y_h + y_p$

$$\boxed{y = C_1 e^{-2t} + C_2 e^{2t} - 2t^2 + t + 1 - 2t\cos(2t)}$$

Exercise 4. (10pt) Find first the general solutions to the following system and afterwards the solution to the initial value problem.

$$\vec{y}' = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \vec{y}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 1 \\ -6 \\ -2 \end{pmatrix}$$

$$\lambda = -1, 3 \qquad E_3 = \begin{pmatrix} -4 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix}$$

$$E_{-1} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{AlgMul} = 3, \ \text{GeoMul} = 1$$

$$A+I \quad \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A+I)^2 \quad \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 16 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$(A+I)^3 \quad \begin{bmatrix} 0 & 0 & 32 & -24 \\ 0 & 0 & 64 & -48 \\ 0 & 0 & 256 & -192 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$

$$C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix}$$

$$C_2 e^{t} \left(\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$C_3 e^{-t} \left(\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$C_4 e^{-t} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_g = C_1 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} + C_2 e^{-t} \left(\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + C_3 e^{-t} \left(\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + C_4 e^{-t} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y(0) = \begin{matrix} C_1 \\ 2C_1 \\ 8C_1 \\ 0 \end{matrix} + \begin{matrix} 0 \\ 0 \\ 3C_2 \\ 4C_2 \end{matrix} + \begin{matrix} 0 \\ 3C_3 \\ 0 \\ 0 \end{matrix} + \begin{matrix} 6C_4 \\ 0 \\ 0 \\ 0 \end{matrix} = \begin{matrix} 3 \\ 1 \\ -6 \\ -2 \end{matrix}$$

$$C_1 + 6C_4 = 3$$
$$2C_1 + 3C_3 = 1$$
$$8C_1 + 3C_2 = -6$$
$$4C_2 = -2$$

$$C_1 = -\frac{9}{16} \quad C_2 = -\frac{1}{2} \quad C_3 = \frac{17}{24} \quad C_4 = \frac{57}{96}$$

$$\boxed{y = -\frac{9}{16} e^{3t} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 0 \end{bmatrix} - \frac{1}{2} e^{-t} \begin{bmatrix} 3t^2 \\ 3t \\ 3 \\ 4 \end{bmatrix} + \frac{17}{24} e^{-t} \begin{bmatrix} 6t \\ 3 \\ 0 \\ 0 \end{bmatrix} + \frac{57}{96} e^{-t} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$16 \times 3 = 48$ $\quad -48 \times 4 = -192$ $\quad \frac{57}{16} \cdot \frac{1}{6} =$ $\quad 16 \times 6 = 96$

Exercise 5. (8pt) Consider the differential equation

$$t^2y'' - (t^2+2t)y' + (t+2)y = 2(e^t - 1) - t(e^t+1), \qquad (t > 0)$$

(1) Show that $y_1 = e^t(2t+1) - (t+1)$ is solutions to the above equation. (4pt)
(Show ALL your calculations in detail for full credit)

$y_1 = e^t(2t+1) - (t+1)$

$y_1' = 2e^t + e^t(2t+1) - 1 = e^t(2t+3) - 1$

$y_1'' = 2e^t + e^t(2t+3) = e^t(2t+5)$

$t^2(e^t(2t+5)) - (t^2+2t)((2t+3)e^t - 1) + (t+2)(e^t(2t+1) - (t+1))$

$t^2e^t(2t+5)$

$\underbrace{e^t(2t^3+5t^2)}_{0} - \underbrace{e^t(2t^3+7t^2+6t)}_{0} + t^2+2t + e^t(2t^2+5t+2) - t^2-3t-2$

$-e^t6t + \underbrace{t^2}+2t + e^t5t + e^t2 \underbrace{-t^2} -3t-2$

$2e^t - te^t - t - 2$

$2e^t - 2 - te^t - t$

$\boxed{2(e^t-1) - t(e^t+1)}$

$y_1 = e^t(2t+1) - (t+1)$

(2) Given that $y_2 = e^t(t+1)+(t-1)$, and $y_3 = e^t(1-t)+(2t-1)$ are also solutions to the above equation, find the general solution to the equation. Justify your answer. (4pt)

$y = y_h + y_p$

$y_1 = 2te^t - t + e^t - 1$

$y_2 = te^t + t + e^t - 1$

$y_3 = \underbrace{-te^t + 2t}\ \underbrace{+e^t - 1}$

Since $y = y_g + y_p$

is the same

$t^2y'' - (t^2+2t)y' + (t+2)y = 0$ general

try $y = Ate^t$:

$y' = Ae^t + Ate^t$

$y'' = Ae^t + Ae^t + Ate^t$

$t^2(Ate^t + 2Ae^t) - (t^2+2t)(Ate^t + Ae^t) + (t+2)(Ate^t)$

$At^3e^t + 2At^2e^t - At^3e^t - 2At^2e^t - At^2e^t - 2Ate^t + At^2e^t + 2Ate^t$

$= 0$

gen soln Cte^t

$y'' - (1+\frac{2}{t})y' + (\frac{1}{t} + \frac{2}{t^2})y = 0$ $\rightarrow \boxed{Cte^t} \leftarrow$

It's the homogenous solution, and you can just tack on particular solutions to have those too

Exercise 6. (9pt) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and consider the system of differential equations $\vec{y}' = A\vec{y}$.

(1) Give the general solution for $\vec{y}' = A\vec{y}$ (5pt)

$\lambda^2 - 4\lambda + 5$

$\lambda = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

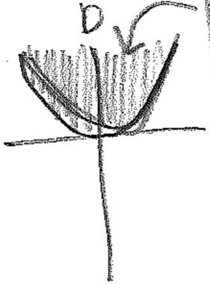
choose $2+i$

$E_{2+i} = \begin{pmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{pmatrix} = \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix} \begin{bmatrix} 1-i \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$c_1 e^{2t}\left(\cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + c_2 e^{2t}\left(\sin t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

$$\boxed{y = e^{2t}\left(c_1 \begin{bmatrix} \cos t + \sin t \\ -\cos t \end{bmatrix} + c_2 \begin{bmatrix} \sin t - \cos t \\ -\sin t \end{bmatrix}\right)}$$

(2) Conclude that the equilibrium point is a spiral. (1pt)



$\boxed{T^2 - 4D < 0}$

must be a spiral,

also, imaginary eigenvalue

(3) Is it a sink or a source? (1pt)

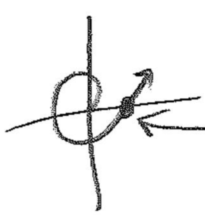
$\boxed{\text{source}}$ since $\alpha > 0$

if eigenvalue $\lambda = \alpha + i\beta$

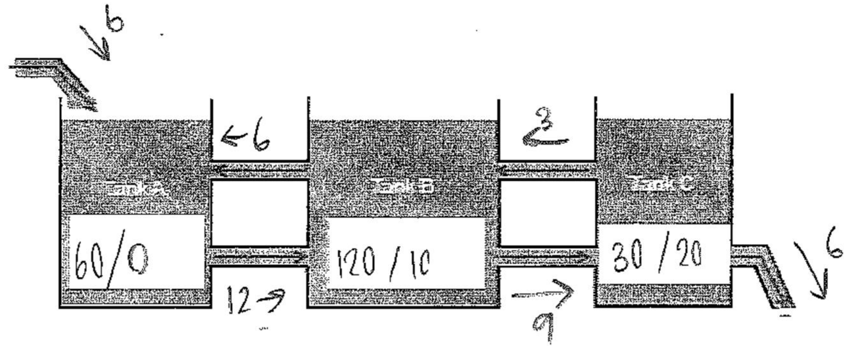
(4) Does the spiral rotate clockwise or counterclockwise? (2pt)

Plug in point $(1,0)$

$y' = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$=$ $\boxed{\text{counterclockwise}}$



$V_A = 6+6-12$

$V_B = 12+3-6-9$ = volumes constant!

$V_C = +9-3-6$

$X = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$

$X(0) = \begin{bmatrix} 0 \\ 10 \\ 20 \end{bmatrix}$

Exercise 7. (9pt)
Consider the above mixing problem with the following data.

- at time $t = 0$ there is 0 lb of salt in tank A, 10 lb of salt in tank B, and 20 lb of salt in tank C.
- Tank A contains initially 60 gallons of solution, Tank B contains initially 120 gallons of solution and Tank C contains initially 30 gallons of solution.
- 6 gal/min of water enters tank A through the top far left pipe.
- The solutions flows at
  - at 6 gal/min through the upper left pipe
  - at 12 gal/min through the lower left pipe
  - at 3 gal/min through the upper right pipe
  - at 9 gal/min through the lower right pipe
- 6 gal/min of solutions leaves tank C through the bottom far right pipe.

$X_A' = \frac{-12 x_A}{60}$

Set up an initial value problem that models the salt content $x_A(t)$ and $x_B(t)$ and $x_C(t)$ in tank A, B, and C at time t (**you do NOT have to solve it!**).

$$X_A' = \frac{-12 X_A}{60} + \frac{6}{120} X_B + 0 X_C$$

$$X_B' = \frac{12 X_A}{60} - \frac{15}{120} X_B + \frac{3}{30} X_C$$

$$X_C' = 0 X_A + \frac{9}{120} X_B - \frac{9}{30} X_C$$

$$X = \begin{bmatrix} X_A \\ X_B \\ X_C \end{bmatrix}$$

$$X' = \begin{bmatrix} -12/60 & 6/120 & 0 \\ 12/60 & -15/120 & 3/30 \\ 0 & 9/120 & -9/30 \end{bmatrix} X$$

$$X(0) = \begin{bmatrix} 0 \\ 10 \\ 20 \end{bmatrix}$$

Always positive so I don't care

$$\begin{array}{r} x^2 - 4x + 3 \\ x-1 \overline{\big) x^3 - 5x^2 + 7x - 3} \\ -(x^3 - x^2) \\ -4x^2 + 7x \\ -(-4x^2 + 4x) \\ 3x - 3 \end{array}$$

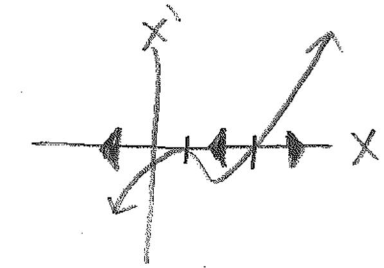
Exercise 8. (8pt)
Consider the differential equation

$$\frac{dx}{dt} = e^x(x^3 - 5x^2 + 7x - 3)$$

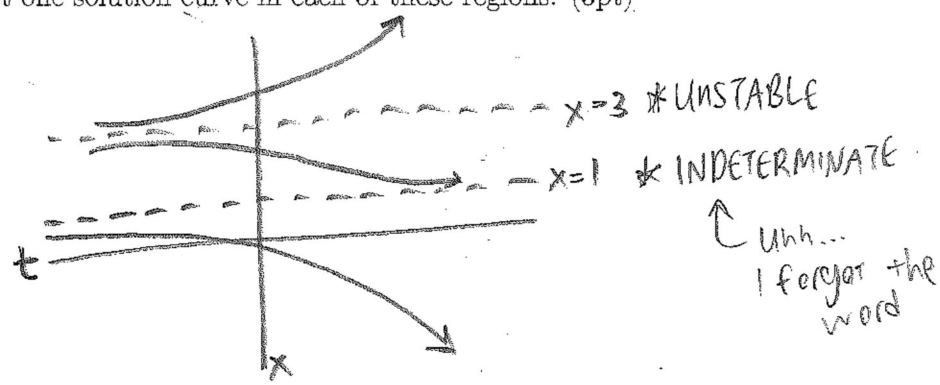
$(x-1)(x-3)(x-1)$

(1) Identify the equilibrium points and sketch the phase line diagram of the equation. (3pt)

equilibrium points = 1, 3



(2) Sketch the equilibrium points on the tx-plane and identify the stable and unstable points. The equilibrium solutions divide the tx-plane into regions. Sketch at least one solution curve in each of these regions. (3pt)



(3) Does there exist a solution of the equation, $x(t)$, satisfying $x(0) = -1$ and $x(2) = 0$? Justify your answer. (2pt)

$\boxed{\text{NO,}}$ by looking at the function $f = \frac{dx}{dt}$, you can tell both f and $\partial f/\partial x$ are continuous so you can apply uniqueness and existance to say no solutions cross other solutions. then notice that solutions trend Downwards in the secton below $X=1$, while $(0,-1) \rightarrow (2,0)$ would imply the solution curve needs to rise. That would be impossible unless this particular solution crossed other downward facing solutions.

A→B impossible

$AX - A = X \quad A + B = 1 \quad -B = 1$
$BX - 2B \quad -A - 2B = 0 \quad B = -1$
$A = 2$

$\int \frac{x}{(x-2)(x-1)}\,dx = \int \frac{1}{t}\,dt$

$\int \frac{2}{x-2}\,dx + \int \frac{-1}{x-1}\,dx = \int \frac{1}{t}\,dt$

$2\ln|x-2| - \ln|x-1| = \ln|t| + C$

$\ln\left|\frac{(x-2)^2}{x-1}\right| = \ln|t| + C$

$\frac{(x-2)^2}{x-1} = te^C$

Exercise 9. (9pt) Consider the differential equation

$$\frac{dx}{dt} = \frac{x^2 - 3x + 2}{tx}$$

(1) Find all constant solutions of the above equation. (4pt)

* points where $\frac{dx}{dt} = 0$ $(x-2)(x-1) =$ $\boxed{x = 1, 2 \text{ where } t \neq 0}$

(2) Argue that the range of the solution to the initial value problem $x(1) = 1.2$ is contained in $(1, 2)$. (3pt)

$x=1$ and $x=2$ are equilibrium solutions.
As long as $t \neq 0$, existance and uniqueness is perserved and that means it's impossible to jump over the equilibrium solutions
So the problem is trapped between $(1,2)$

(3) Can you apply the existence theorem to the initial value problem $y(0) = 5$? (1pt) Justify your answer. (1pt)

$\boxed{\text{NO}}$, $\frac{dx}{dt} = \frac{x^2-3x+2}{tx}$ is not defined at $t=0$,
you can not divide by 0.

Exercise 10. (9pt)

(1) Find the value of the constant b and m such that the following equation is exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$. (3pt)

$$2(x + xy^2) + b(x^m y + y^2)\frac{dy}{dx} = 0$$

$2(x+xy^2)dx + b(x^m y + y^2)dy = 0$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$4xy = bmx^{m-1}y$

$m=2$
$b=2$

(2) Solve the equation using the value of b and m you obtained in part (a). (6pt)

$(2x+2xy^2)dx + 2(x^2y+y^2)dy = 0$

$\int 2x+2xy^2 dx = x^2+x^2y^2+\phi(y)$

$\phi'(y) = 2x^2y+2y^2 - \frac{d}{dy}(x^2+x^2y^2) = 2x^2y+2y^2-2x^2y = 2y^2$

$\phi = \int 2y^2 dy = \frac{2}{3}y^3 + C$

$\boxed{F(x,y) = x^2+x^2y^2+\frac{2}{3}y^3 = C}$

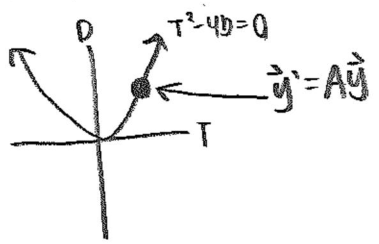
Exercise 11. (11pt) Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$

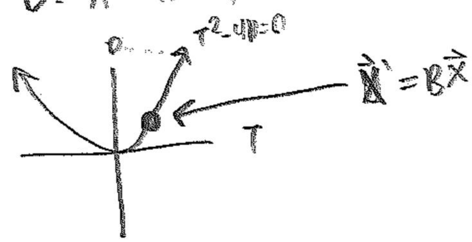
$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(1) Determine where in the trace-determinante plane the system $\vec{y}' = A\vec{y}$ and $\vec{x}' = B\vec{x}$ fit. (3pt)

A: $\lambda^2 - 6\lambda + 9$



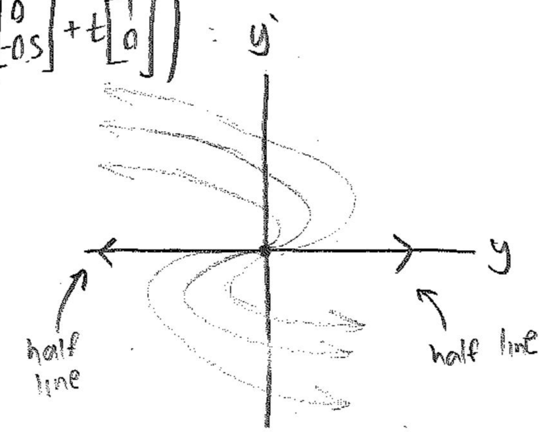
B: $\lambda^2 - 4\lambda + 4$



(2) Find all of the half line solutions for the system $\vec{y}' = A\vec{y}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

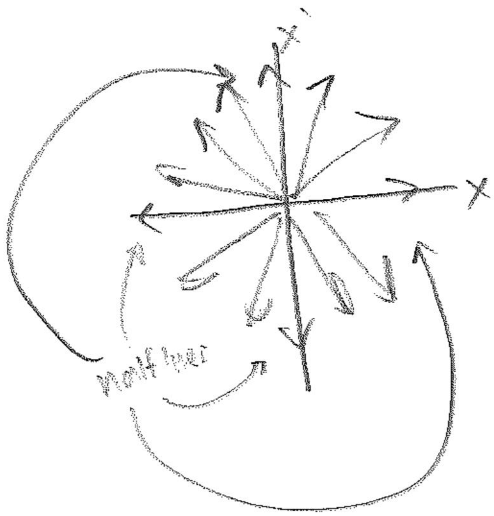
$$y = C_1 e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t}\left(\begin{bmatrix} 0 \\ -0.5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



(3) Find all of the half line solutions for the system $\vec{x}' = B\vec{x}$. (2pt) Sketch them into the y_1, y_2 coordinate system (2pt).

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Everything is a half line solution!

Exercise 12. (5pt)

(1) Consider the second order equation $y'' + 3t^2y' - \cos(t)y = -3e^t$. Write this equations as a planar system of first-order equations. (2pt)

$y' = v \quad y'' = v'$

$v' + 3t^2 v - \cos(t) y = -3e^t$

$$\boxed{v' = -3t^2 v + \cos(t) y - 3e^t}$$

(2) Consider more generally an n-order equation $y^{(n)} = F(t, y, \ldots, y^{(n-1)})$. How can you write this as a system of first-order equations? (3pt)

say $y_0 = y$

$y_1 = y'$

$y_2 = y'' = y_1'$

$y_3 = y''' = y_2'$

$\cdots$

$y_{n-1} = y^{(n-1)} = y_{n-2}'$

$c(t) +$

$$\boxed{y^{(n)} = F(t, y_0, y_1, y_2, \ldots y_{n-1})}$$

Extra page